TRISTAN BUCKMASTER (NEW YORK UNIVERSITY), MÜNSTER MIDTERM CONFERENCE, 27 MARCH 2024

Singularities in Fluids

Self-similar Analysis, Computer Assisted Proofs and Neural Networks

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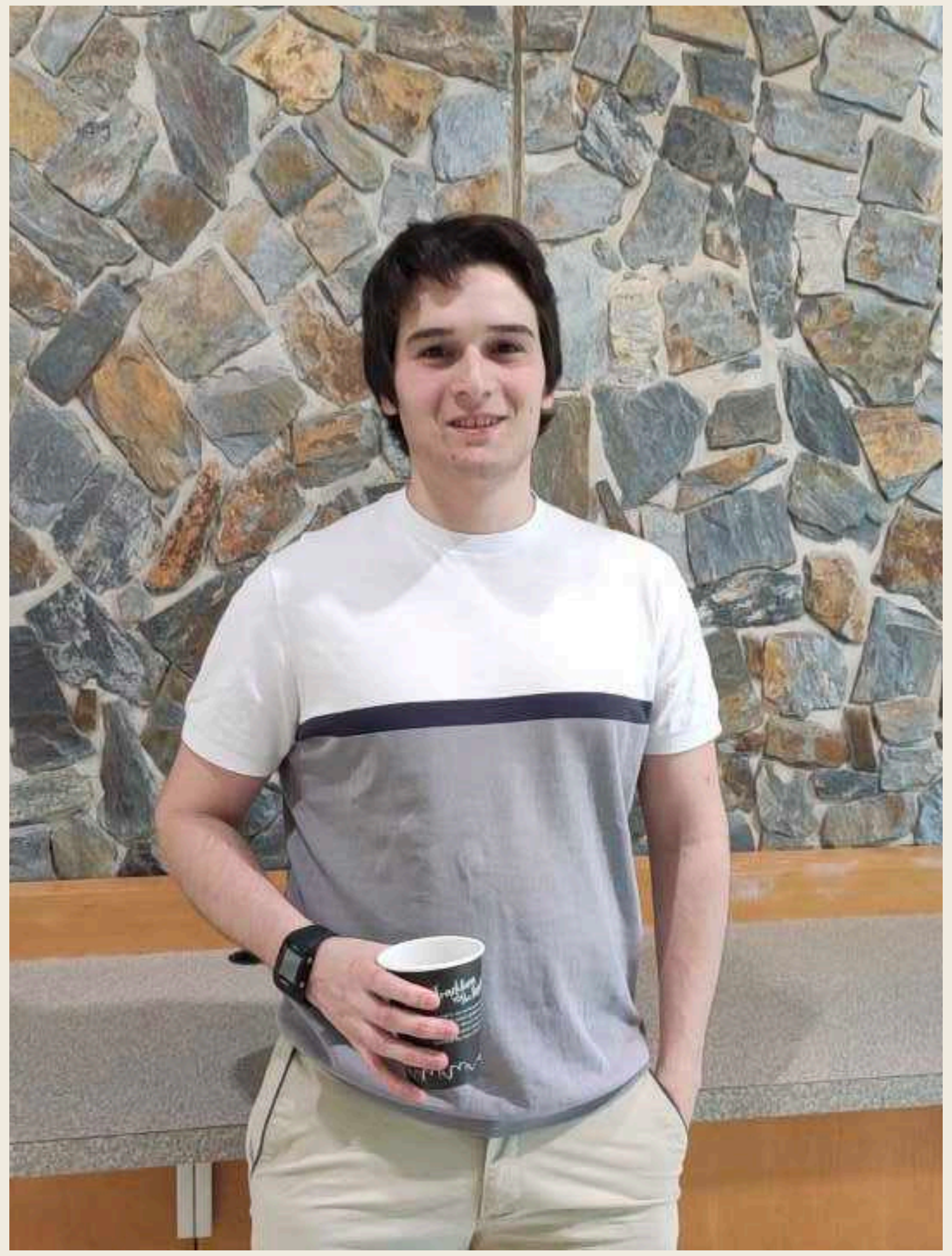
Rising stars



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Postdoc Geosciences, Princeton,
Future postdoc, Courant, NYU

The singularity problem

The incompressible Euler/Navier Stokes equation:

$$\partial_t v + v \cdot \nabla v + \nabla p - \nu \Delta v = 0, \qquad \operatorname{div} v = 0,$$

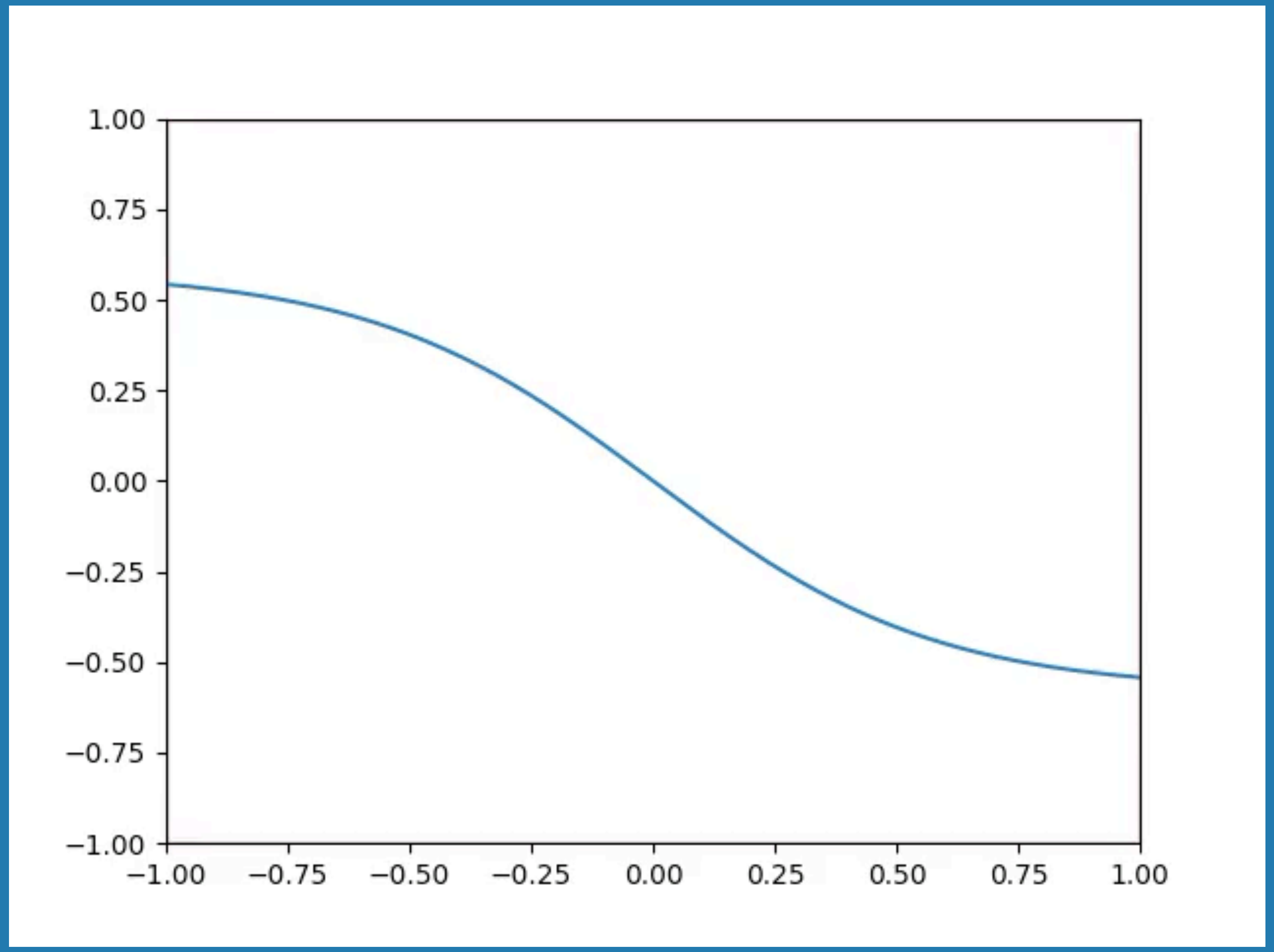
for viscosity $\nu > 0$, velocity v and pressure p.

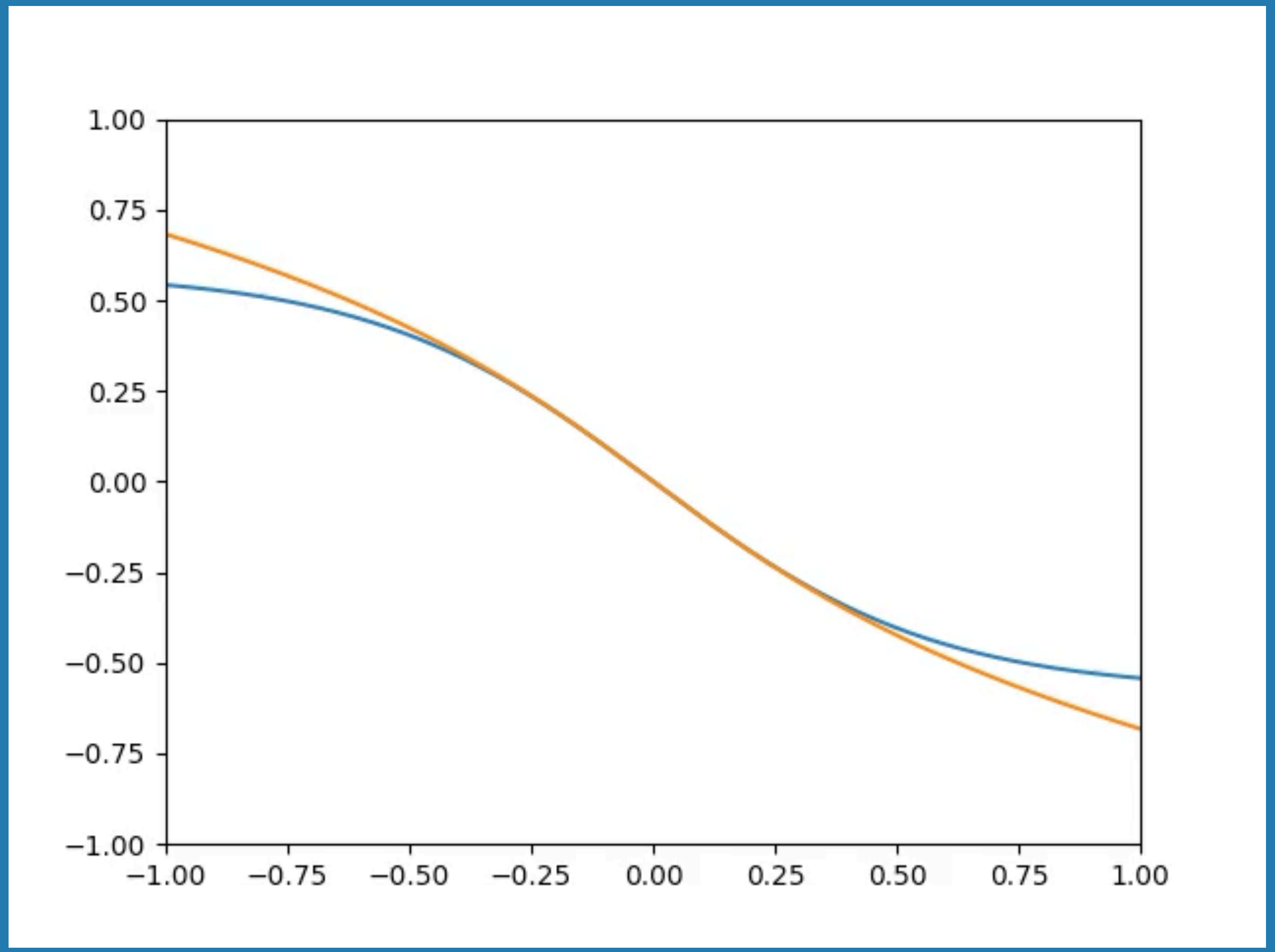
Open Problem: For the 3-D Euler/Navier-Stokes, does there exist smooth initial data $v(0,t) = v_0$ leading to the formation of a singularity in finite time?

Eligindi '19 answered the question for *non-smooth* $C^{1,\alpha}$ Euler initial data (cf. *Elgindi Ghoul Masmoudi '19*). An analogous result was proven by *Chen Hou '19* for the case of Euler with cylindrical boundary.

Blow up for Euler with cylindrical boundary from smooth initial data was proven by *Chen-Hou '22/'23*!

Self-similar analysis:
Shock like singularities as a model





Toy Problem: Burgers' equation

Consider the Burgers' equation

$$\partial_t u + u\partial_x u = 0 \text{ where } u(x,0) = u_0 \,.$$

Let η^{y_0} be the characteristic induced by u, starting at y_0:

$$\eta^{y_0}(t) = y_0 + u_0(y_0)t \,.$$

Then, one can solve Burgers' via characteristics

$$u \circ \eta^{y_0} = u_0(y_0) \,.$$

Taking a derivative of u and following characteristics

$$\frac{d}{dt}(u_x \circ \eta^{y_0}) = -u_x^2 \circ \eta^{y_0}\,.$$

Thus, if $u_0'(y_0) = \alpha$, then

$$u_x \circ \eta^{y_0} = \frac{\alpha}{\alpha t + 1}\,.$$

If $\alpha < 0$, then a singularity forms at time $-\frac{1}{\alpha}$.

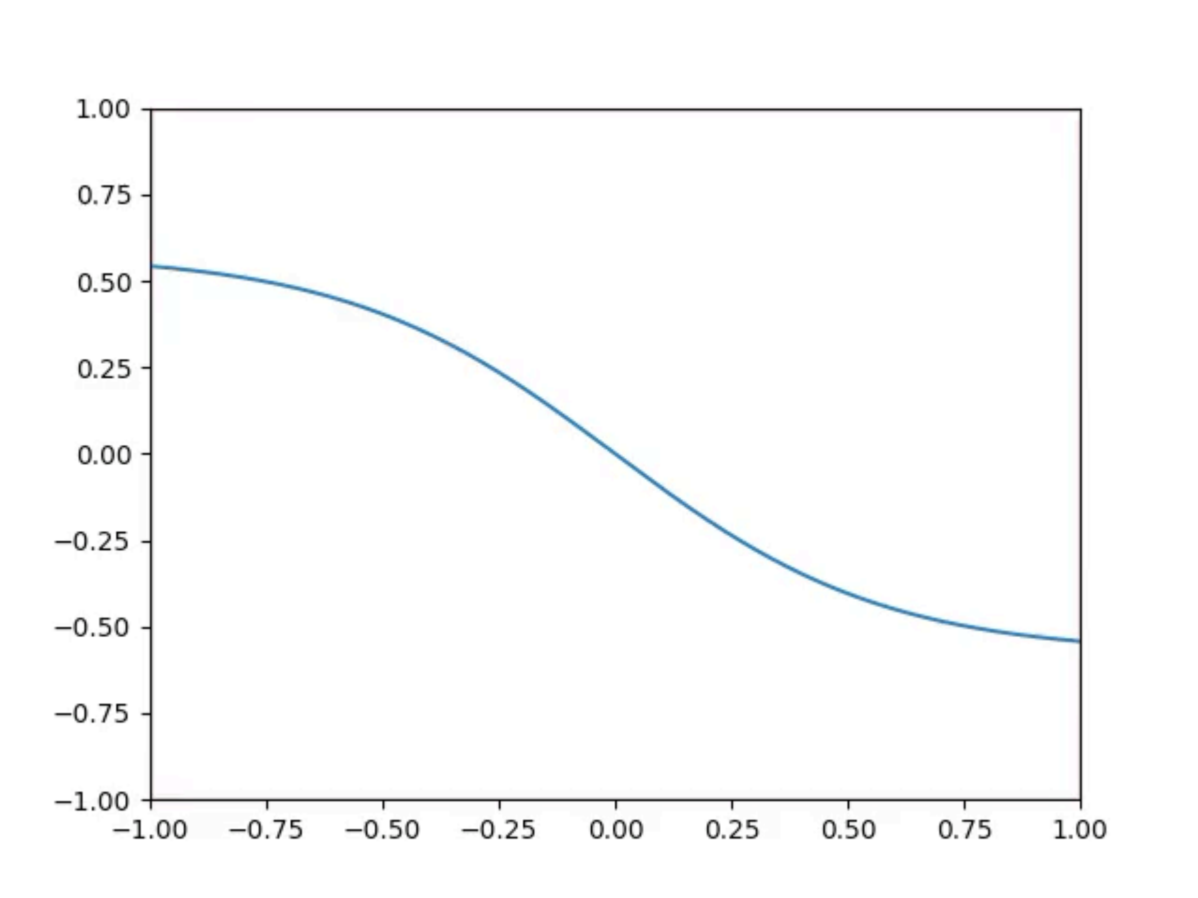
Self-similar analysis simplifies the search for singularities by extracting the *blow-up profile*, describing the behavior at the location and time of the singularity.

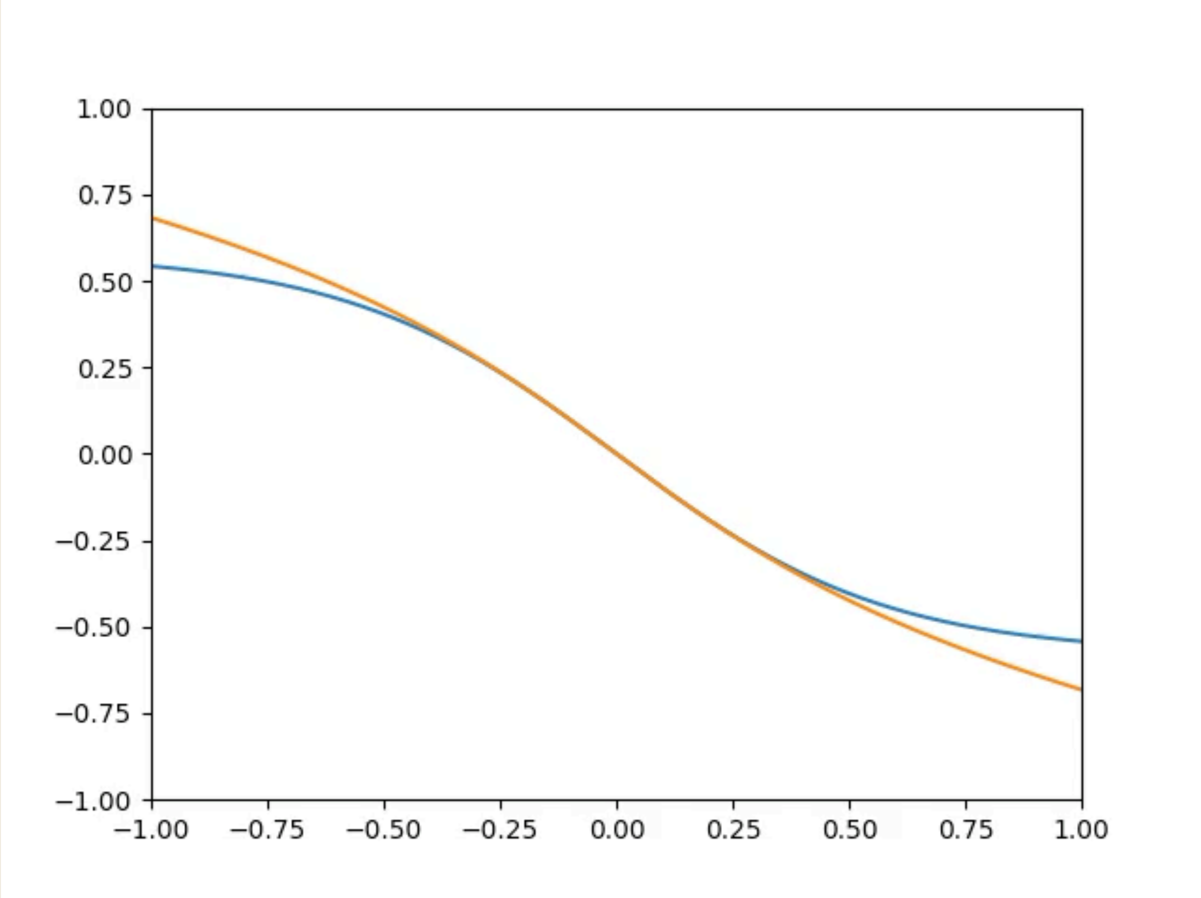
For the Burgers' equation $u_t + uu' = 0$. Consider the ansatz

$$u(x,t) = (1-t)^{\lambda} U\Big(\underbrace{\frac{x}{(1-t)^{1+\lambda}}}_{y}\Big).$$

The PDE reduces to the ODE

$$-\lambda U + \big((1+\lambda)y + U\big)\, U' = 0.$$





The self-similar equation

$$-\lambda U + \big((1+\lambda)y + U\big)\, U' = 0,$$

has implicit solution

$$y = -\,U - CU^{1+\frac{1}{\lambda}} \text{ for any constant } C.$$

For U to be smooth $\Longrightarrow 1 + \dfrac{1}{\lambda} \in \mathbb{N}$, and U to defined globally $\Longrightarrow 1 + \dfrac{1}{\lambda}$ odd and $C > 0$. We are left with

$$\lambda = \frac{1}{2i+2} \text{ for } i = 0,1,2,\ldots$$

The corresponding solutions are odd.

For $i = 0$, i.e., $\lambda = \frac{1}{2}$, setting $C = 3! = 6$, the explicit solution $\bar{U}$ is

$$\bar{U}(y) = \left(-\frac{y}{2} + \left(\frac{1}{27} + \frac{y^2}{4}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} - \left(\frac{y}{2} + \left(\frac{1}{27} + \frac{y^2}{4}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}$$

$\bar{U}$ is stable modulo the symmetries of Burgers' equation: Suppose u solves Burgers' then define $\tilde{U}$ by

$$u(x,t) = (1-t)^{\lambda}(\bar{U}(y) + \tilde{U}(y,s))$$

for $s = -\log(T-t)$. Then, $\tilde{U}$ solves

$$\partial_s \tilde{U} + \underbrace{(1+\bar{U}')\tilde{U} + ((1+\lambda)y + \bar{U})\tilde{U}'}_{-\mathscr{L}\tilde{U}} = \text{(nonlinear terms in } \tilde{U})$$

All eigenvalues with positive real part of $\mathscr{L}$ are generated by the symmetries of Burgers' (*Collot-Ghoul-Masmoudi '18*). Such a self-similar solution is said to be *stable*.

Computer assisted proof by example:
Implosion for compressible fluid

```
  arb_sub_si(aux,gatilde,2,prec);
  arb_mul(aux,aux,rtilde,prec);
  arb_add(res,res,aux,prec);

  arb_div_si(res,res,4,prec);

  arb_clear(aux);
  arb_clear(R1_desing);
  return;
}


// We use that W0 - 2*(1-rtilde)/gatilde - rtilde/(rtilde-1)
// = gatilde/4*(2 R1desing_twice + R1desing + rtilde)
void arb_desingularize_W0_twice(arb_t res, arb_t rtilde, arb_t gatilde, slong prec, int& flag){
  arb_t R1_desing, R1_desing_twice;
  arb_init(R1_desing); arb_init(R1_desing_twice);

  arb_desingularize_R1_over_gatilde(R1_desing,rtilde,gatilde,prec,flag);
  arb_desingularize_R1_over_gatilde_twice(R1_desing_twice,rtilde,gatilde,prec,flag);

  arb_mul_si(res,R1_desing_twice,2,prec);
  arb_add(res,res,R1_desing,prec);
  arb_add(res,res,rtilde,prec);

  arb_div_si(res,res,4,prec);

  arb_clear(R1_desing); arb_clear(R1_desing_twice);
  return;
}
```

Implosion Setup

Isentropic, spherically symmetric Euler

$$\partial_t u + u\partial_R u + \frac{1}{\gamma\rho}\partial_R \rho^\gamma = 0 \quad \text{and} \quad \partial_t \rho + \frac{1}{R^2}\partial_R(R^2\rho u) = 0.$$

The self-similar ansatz:

$$u(R,t) = (1+\lambda)\frac{R}{T-t}U(\log(\frac{R}{(T-t)^{1+\lambda}})) \text{ and } \sigma(R,t) = (1+\lambda)\alpha^{-\frac{1}{2}}\frac{R}{T-t}S(\log(\frac{R}{(T-t)^{1+\lambda}})),$$

where $\sigma = \frac{1}{\alpha}\rho^\alpha$ is the rescaled sound speed, $\alpha = \frac{\gamma-1}{2}$. Setting $\xi = \log(\frac{R}{(T-t)^{1+\lambda}})$ leads to the autonomous ODE

$$\frac{dU}{d\xi} = \frac{N_U(U,S)}{D(U,S)}, \quad \text{and} \quad \frac{dS}{d\xi} = \frac{N_S(U,S)}{D(U,S)}.$$

Merle Raphaël Rodnianski Szeftel '19

For a.e. $\gamma > 1$, there exists a countably infinite sequence of self-similar solutions to isentropic Euler. The velocity and density blow up at the origin.

The existence of *non-smooth* imploding shock wave solutions is a classical result of *Guderley '42*.

Compressible Navier-Stokes

Isentropic 3D compressible Navier-Stokes with constant viscosity:

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p(\rho) - \mu_1 \Delta u - (\mu_1 + \mu_2) \nabla \mathrm{div}\, u = 0\,,$$
$$\partial_t \rho + \mathrm{div}(\rho u) = 0\,,$$

for $\mu_1 \geq 0$ and $2\mu_1 + \mu_2 \geq 0$.

Merle Raphaël Rodnianski Szeftel et al. '19: there exists imploding solutions to NS for a.e. $1 < \gamma < \frac{2+\sqrt{3}}{\sqrt{3}}$ with mildly decaying density.

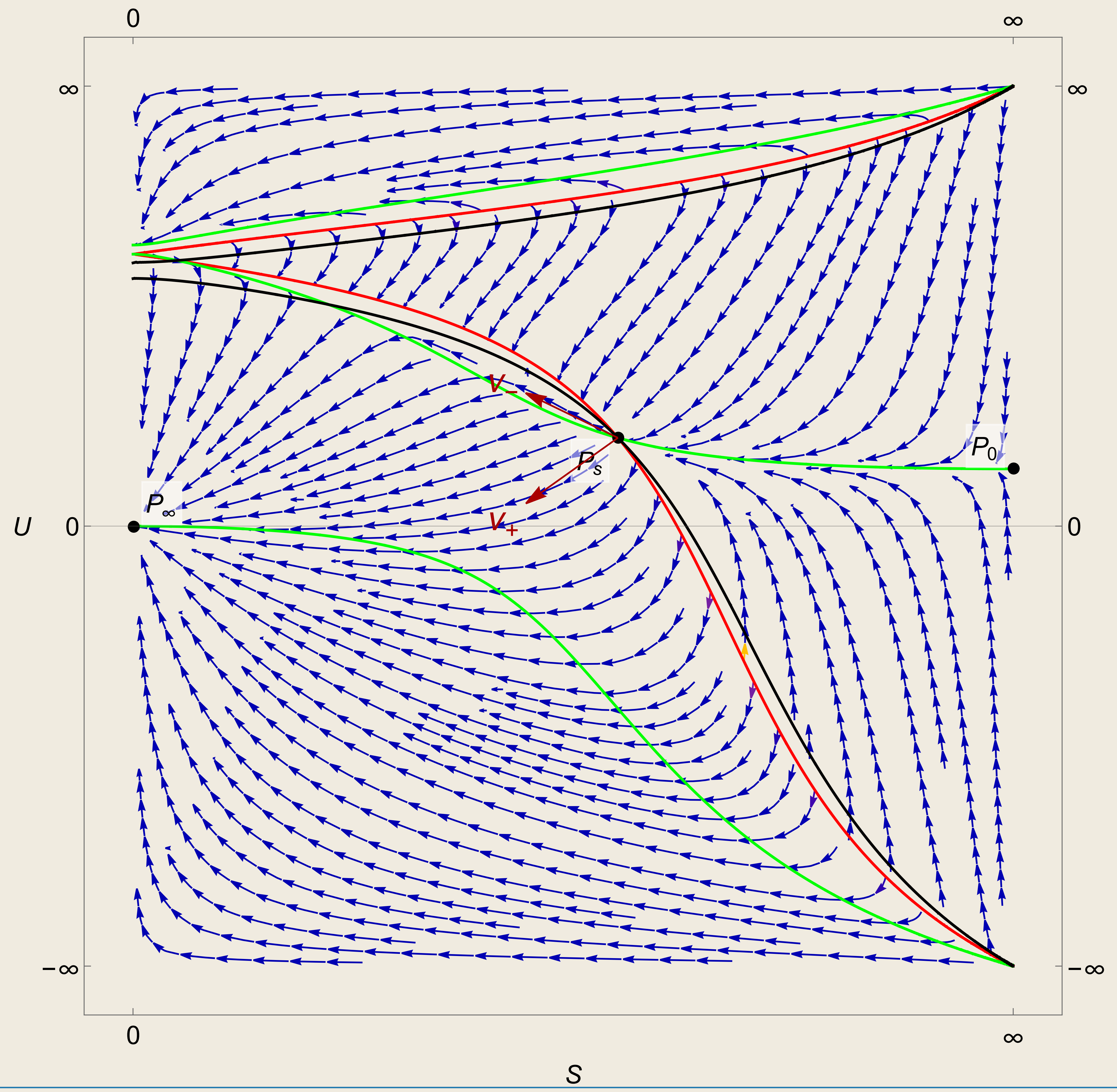
B Cao-Labora Gómez-Serrano '22

- Do imploding solutions for Euler exist for all $\gamma > 1$?
- Can one construct imploding solutions to the Navier-Stokes equation with initial density constant at infinity?

Main result:

- There exists smooth self-similar imploding solutions for all $\gamma > 1$.
- For the case $\gamma = \frac{7}{5}$ (diatomic gas, e.g. oxygen, hydrogen, nitrogen), there exists a countably infinite sequence of imploding solutions.
- Simplified proofs of linear stability and non-linear stability.
- Asymptotically self-similar imploding solutions to NS for $\gamma = \frac{7}{5}$ for initial density constant at infinity.

Phase portrait of ODE



Toy Problem: Barrier Argument

Consider the autonomous ODE:

$$\frac{dx}{dt} = \frac{y + x^2}{(x+4)^2} = \frac{N_1}{D_1} \qquad \frac{dy}{dt} = \frac{-x - y^2}{(x - y + 4)^2} = \frac{N_2}{D_2}.$$

Suppose we want to see if the curve

$$r(t) = \left(t^3, t - 2t^2\right) \text{ for } t \in [0,1],$$

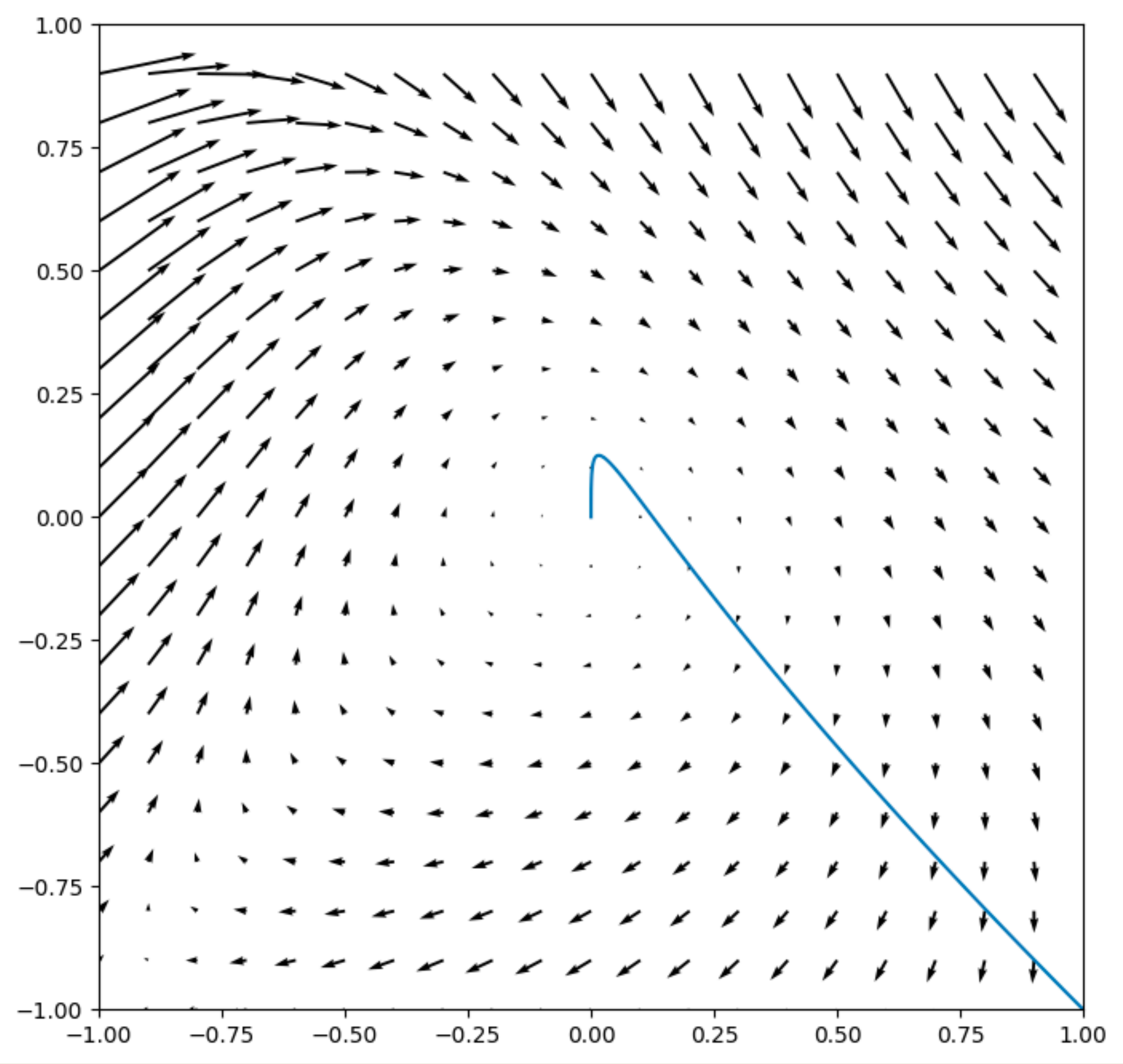
acts as a barrier for the ODE. In particular, we want to show

$$\dot{r}^{\perp}(t) \cdot \left(\frac{N_1}{D_1}, \frac{N_2}{D_2}\right)\Big|_{(x,y)=r(t)} \leq 0 \text{ for all } t \in [0,1],$$

which is equivalent to showing

$$\dot{r}^{\perp}(t) \cdot \left(N_1 D_2, N_2 D_1\right)|_{(x,y)=r(t)} \leq 0,$$

for all $t \in [0,1]$.



Expanding the condition

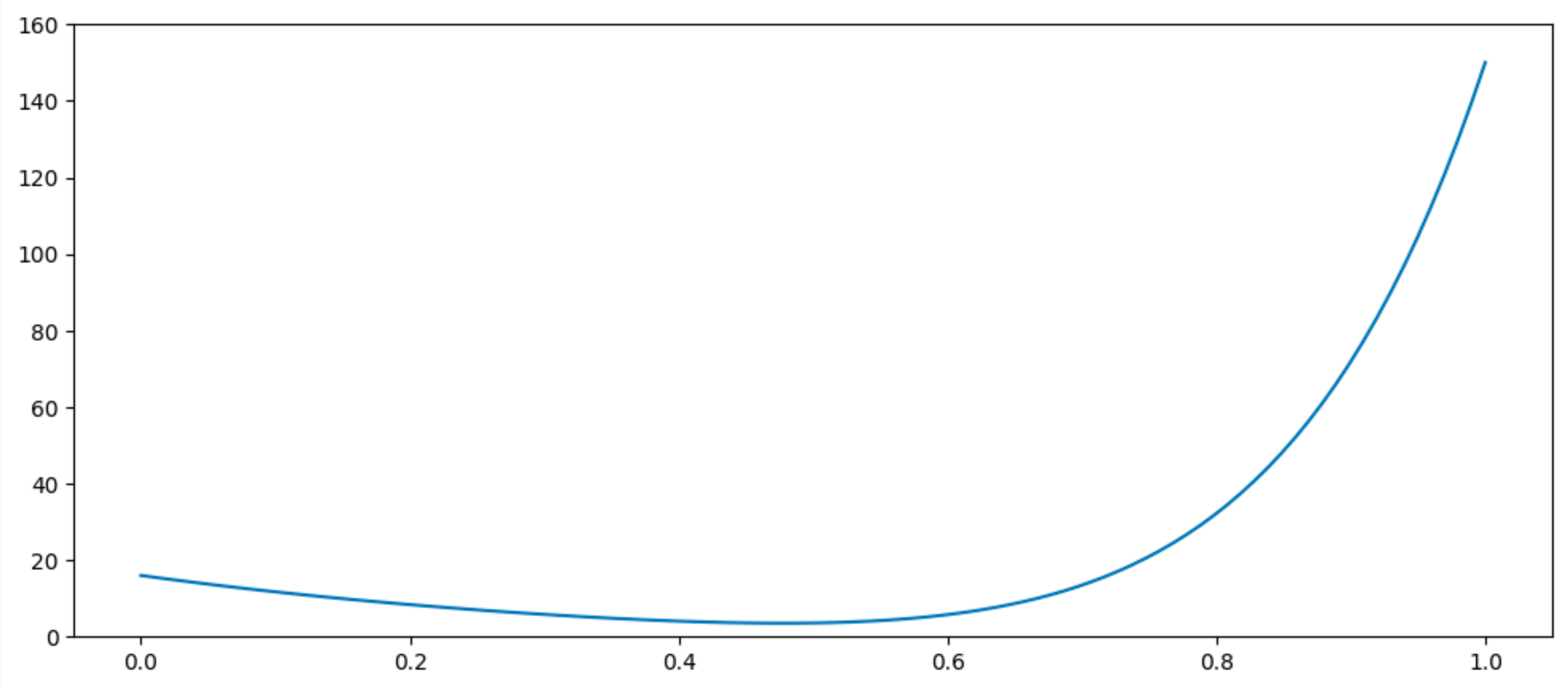
$$\dot{r}^{\perp}(t) \cdot \left(N_1 D_2, N_2 D_1\right) |_{(x,y)=r(t)} \leq 0 \text{ for } t \in [0,1],$$

in t leads to the 13th order polynomial condition

$$-3t^4 \left(t^3+4\right)^2 (t(4t-3)+1) + \frac{1}{2}(t-2)t\left(t^5-2t+1\right)(t(t(t+2)-1)+4)^2 \leq 0 \text{ for } t \in [0,1],$$

which is equivalent (after dividing by $-t$) to checking

$$3t^3 \left(t^3+4\right)^2 (t(4t-3)+1) - \frac{1}{2}(t-2)\left(t^5-2t+1\right)(t(t(t+2)-1)+4)^2 \geq 0 \text{ for } t \in [0,1].$$



Interval arithmetic

Replace arithmetic operators $\{ + , - , \times , \div \}$ acting on $\mathbb{R}$ with interval arithmetic operators $\{[+], [-], [\times], [\div]\}$ acting on intervals:

$$5 \pm 2^{-4} \; [+] \; 3 \pm 2^{-7} = 8 \pm 2^{-3}$$
$$5 \pm 2^{-4} \; [\times] \; 3 \pm 2^{-7} = 15 \pm 2^{-2}$$

For this example, we choose radii of powers of two.

Interval Arithmetic SageMath implementation

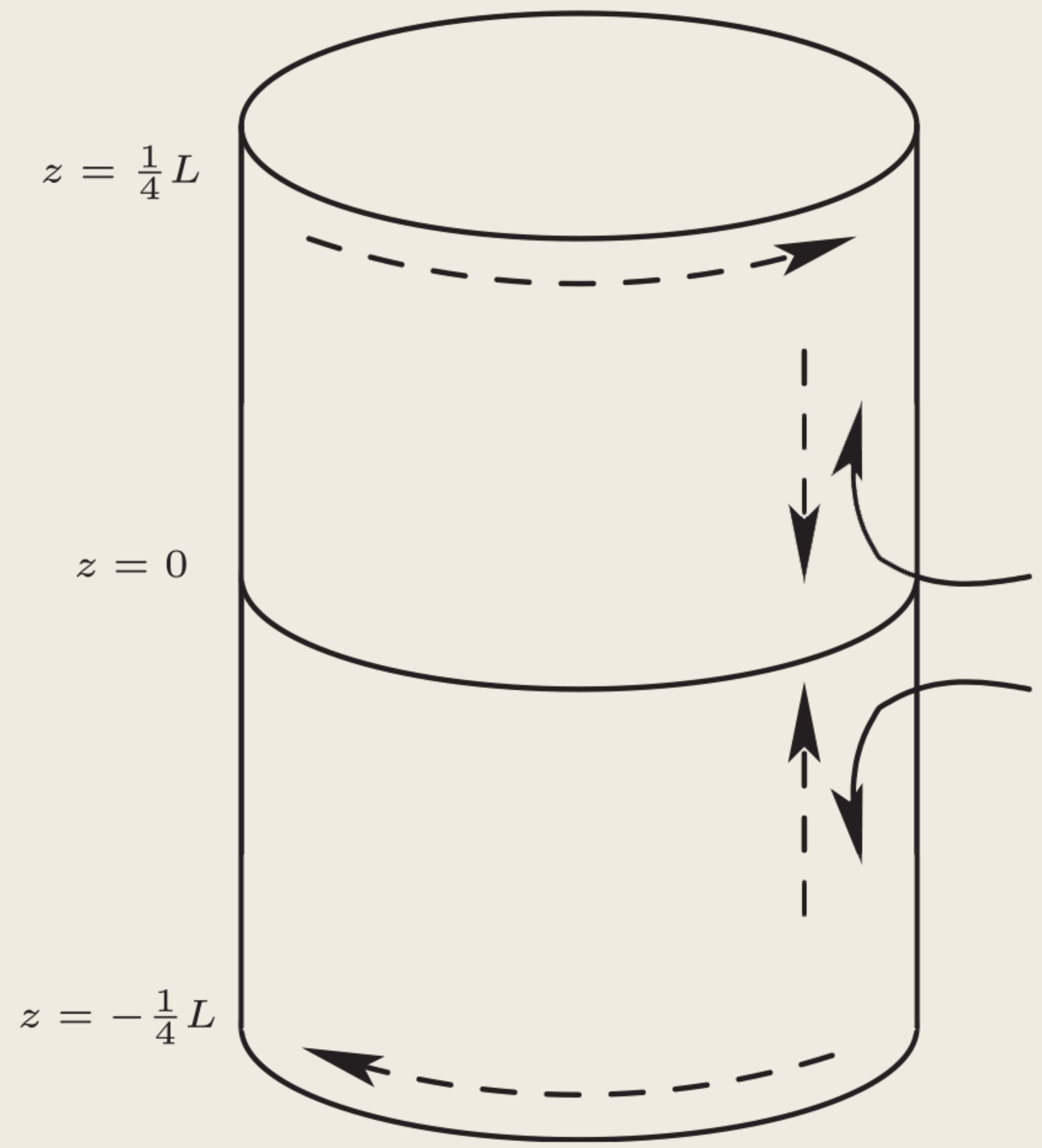
```
def check_positivity(divisions):
    # declare t to be a symbolic variable
    t = var('t')

    # define the polynomial
    f(t) = 3*t^3*(4 + t^3)^2*(1 + t*(-3 + 4*t)) - (1/2)*(-2 + t)*(1 - 2*t + t^5)*(4 + t*(-1 + t*(2 + t)))^2

    # check if polynomial is positive on each subinterval
    for i in range(divisions):
        # define the interval
        interval = RBF(RIF(i/divisions,(i+1)/divisions))
        # evaluate the polynomial on the interval
        check = RBF(f(interval))
        # print the midpoint, radius, and positivity
        print (check.mid(), "+/-",check.rad(), check>0)
```

```
check_positivity(10)

13.7319096680717 +/- 2.4033513 True
9.94519217465356 +/- 2.2604430 True
7.01329109072685 +/- 2.3163407 True
4.83017592207281 +/- 2.6566253 True
3.60764251775374 +/- 3.4963810 True
4.21019282599796 +/- 5.2965599 False
8.65928043033235 +/- 8.8737563 False
20.9688498675823 +/- 15.087557 True
48.5618811141393 +/- 27.722346 True
104.646729755676 +/- 49.986929 True
```

```
check_positivity(13)

14.2320562004355 +/- 1.8402061 True
11.1498422010209 +/- 1.7248105 True
8.61055964292719 +/- 1.6924691 True
6.53495632964663 +/- 1.7562538 True
4.89928321978891 +/- 1.9547040 True
3.80904861562007 +/- 2.3609177 True
3.58923339843750 +/- 3.0933264 True
4.90733564216300 +/- 4.4653752 True
8.95287286089665 +/- 6.6677609 True
17.7064640258238 +/- 10.041640 True
34.3443711368435 +/- 15.858239 True
63.8416146936019 +/- 25.182072 True
113.859875062410 +/- 39.538814 True
```


Physics-Informed Neural Networks:
A novel approach to finding self-similar singularities

Luo-Hou Scenario



Consider incompressible Euler in the exterior of a cylindrical boundary $r \geq 1$.

Luo-Huo '14 gave compelling numerical evidence for blow-up in this setting, suggestive of asymptotic self-similar scaling. See also *Childress '87* and *Pumir Siggia '92*.

A rigorous proof of blow-up from smooth initial data was proven by *Chen-Hou '22/ '23*!

Self-similar blow up for 2-D Boussinesq

The 2-D Boussinesq equations:

$$\partial_t u + u \cdot \nabla u + \nabla p = (0,\theta), \quad \operatorname{div} u = 0 \quad \text{and} \quad \partial_t \theta + u \cdot \nabla \theta = 0.$$

Self-similar ansatz:

$$u = (1-t)^{\lambda} U(y), \quad \theta = (1-t)^{-1+\lambda} \Theta(y), \quad \text{for } y = \frac{(x_1, x_2)}{(1-t)^{1+\lambda}},$$

which lead to

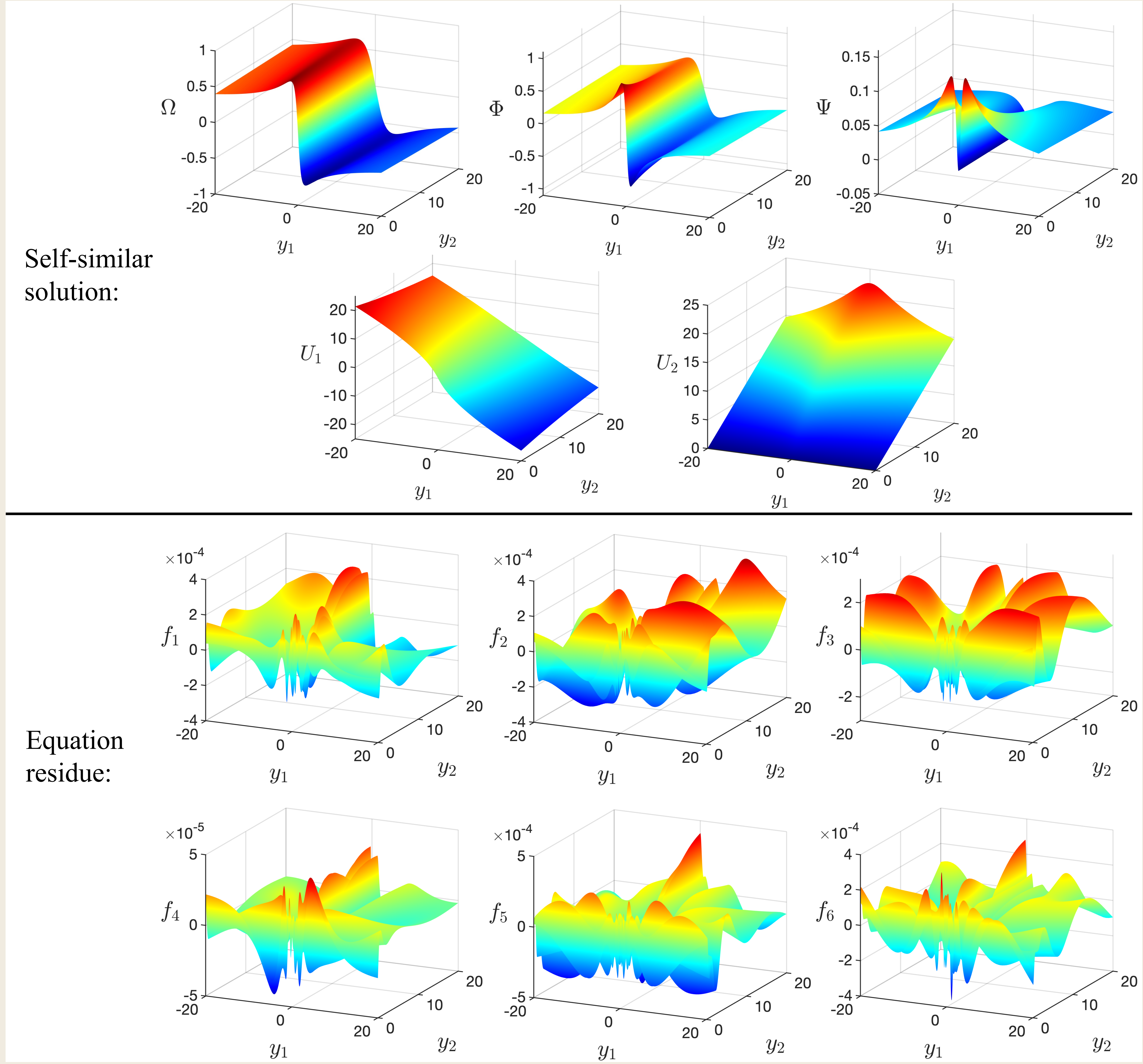
$$-\lambda U + ((1+\lambda)y + U) \cdot \nabla U + \nabla P = (0, -\Theta),$$

$$(1-\lambda)\Theta + ((1+\lambda)y + U) \cdot \nabla \Theta = 0, \quad \text{and} \quad \operatorname{div} U = 0.$$

A *nice* smooth solution implies blow up for Boussinesq.

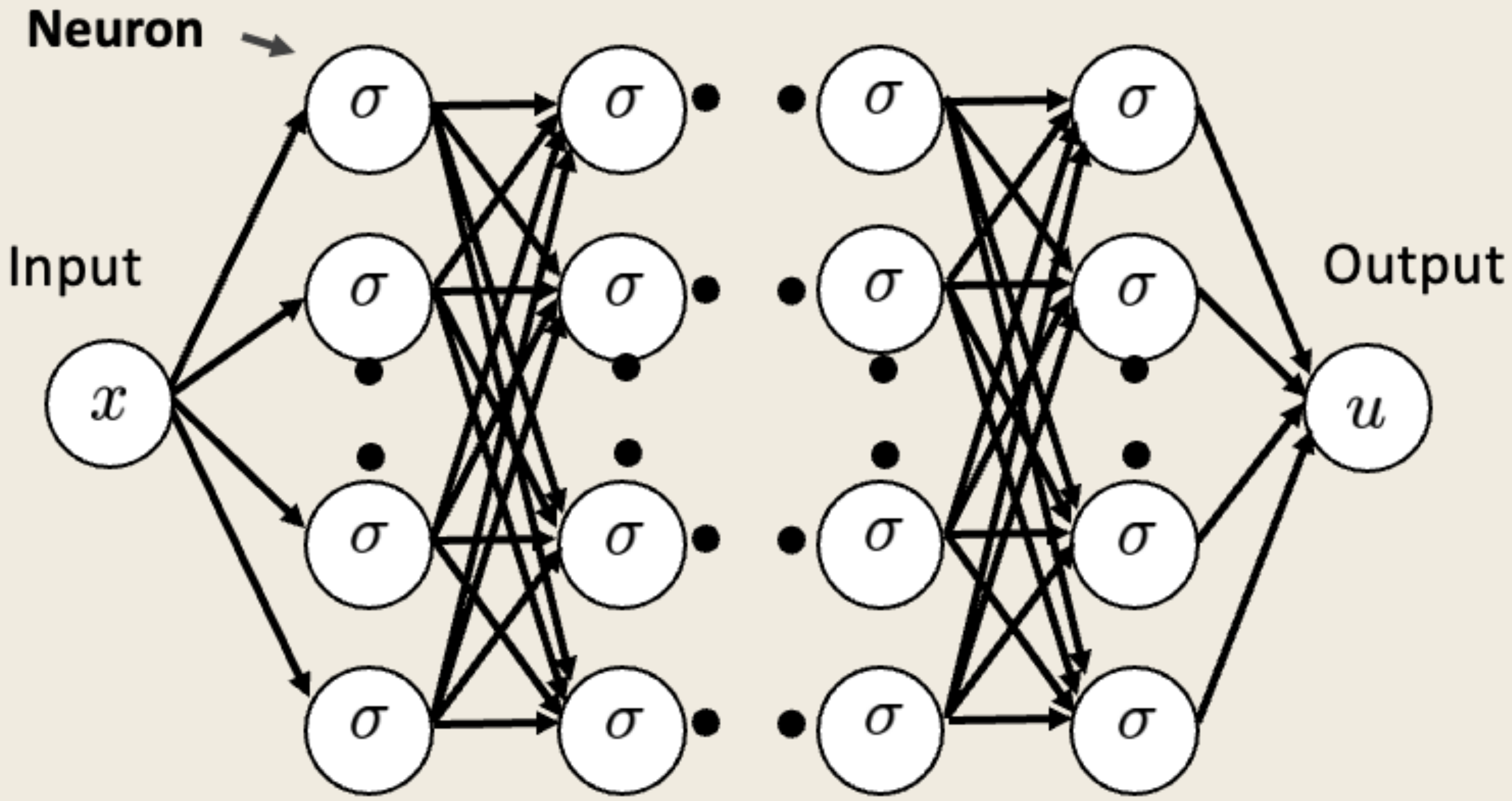
Self-similar Euler = Self-similar Boussinesq + decaying terms.

Wang-Lai-Gomez-Serrano-B ’23 PRL



Neural Network vs Fourier Series

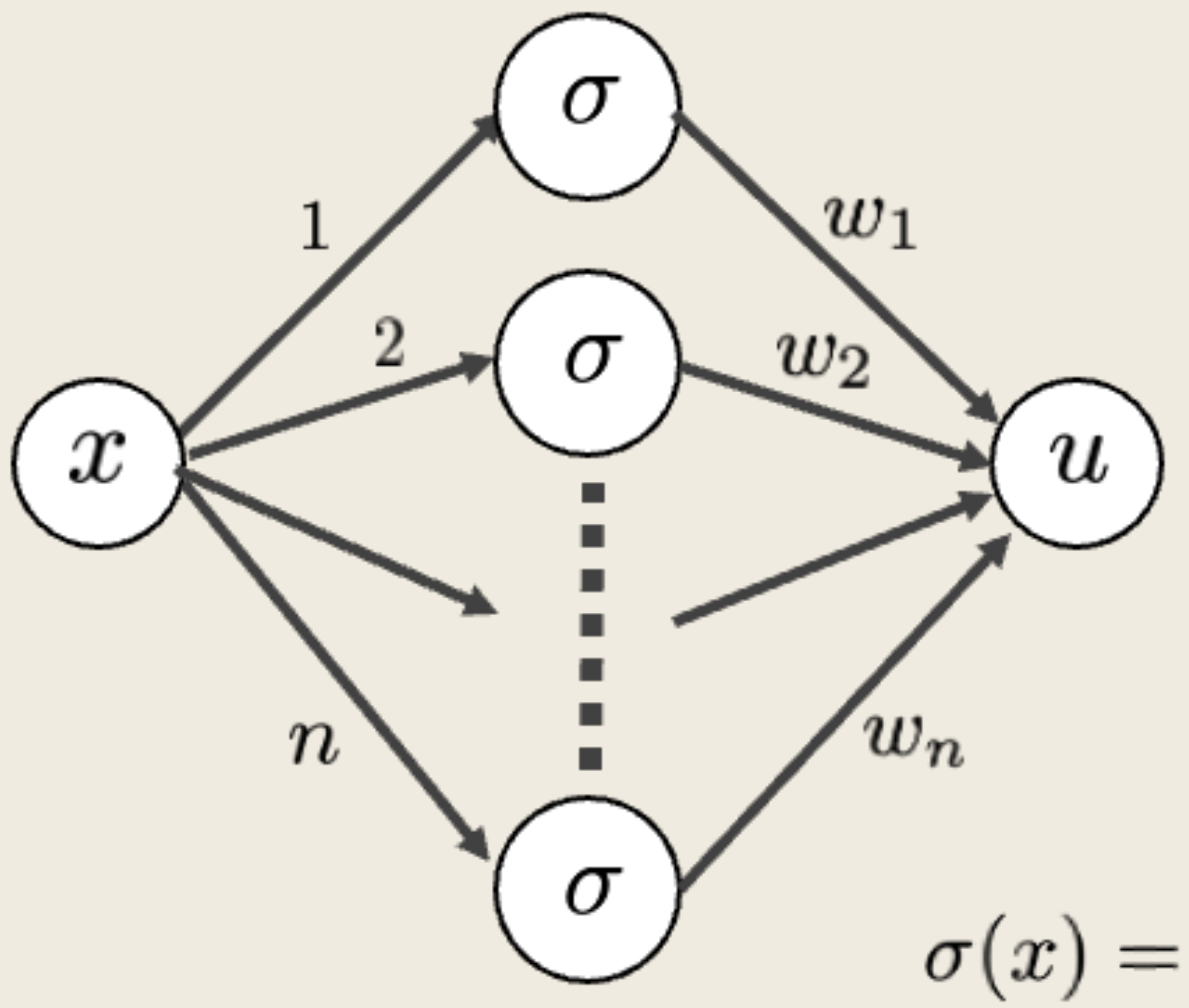
Fully-connected Neural network



$$u(x) = \sum_{j=1} w_{lk}^{(n)} \sigma \left(\dots \sigma \left(\sum_{i=1} w_{ji}^{(1)} \sigma \left(w_i^{(0)} x + b_i^{(0)} \right) + b_j^{(1)} \right) \dots \right) + b_l^{(n)}$$

$\mathbf{w}$: weights $\mathbf{b}$: biases $\sigma(x)$: activation function

Fourier series: $\sin(nx)$



$\sigma(x) = \sin(x)$

Universal function approximator

Hornik et. al. (1989), *Neural Netw.* **2**

Córdoba-Córdoba-Fontelos (CCF) equation

For Córdoba-Córdoba-Fontelos (CCF) equations are

$$\omega_t - u\omega_x - \omega u_x + \mu(-\Delta)^{\alpha/2}\omega = 0 \text{ where } u = \int_0^x (H\omega)(s)\, ds = \Lambda^{-1}\omega\,.$$

Assume the self-similar ansatz:

$$\omega = \frac{1}{1-t}\Omega\left(\frac{x}{(1-t)^{1+\lambda}}\right).$$

With the change of coordinates $y = \frac{x}{(1-t)^{1+\lambda}}$, we obtain the self-similar equations

$$\Omega + ((1+\lambda)y - U)\partial_y\Omega - \Omega\partial_y U - \mu e^{(\alpha(1+\lambda)-1)s}(-\Delta)^{\alpha/2}\omega = 0 \quad \text{where } U = \Lambda^{-1}\Omega.$$

Open Problem

(*Córdoba Córdoba Fóntelos* '05, *Li Rodrigo* '08, *Dong* '08, *Kiselev* '10):

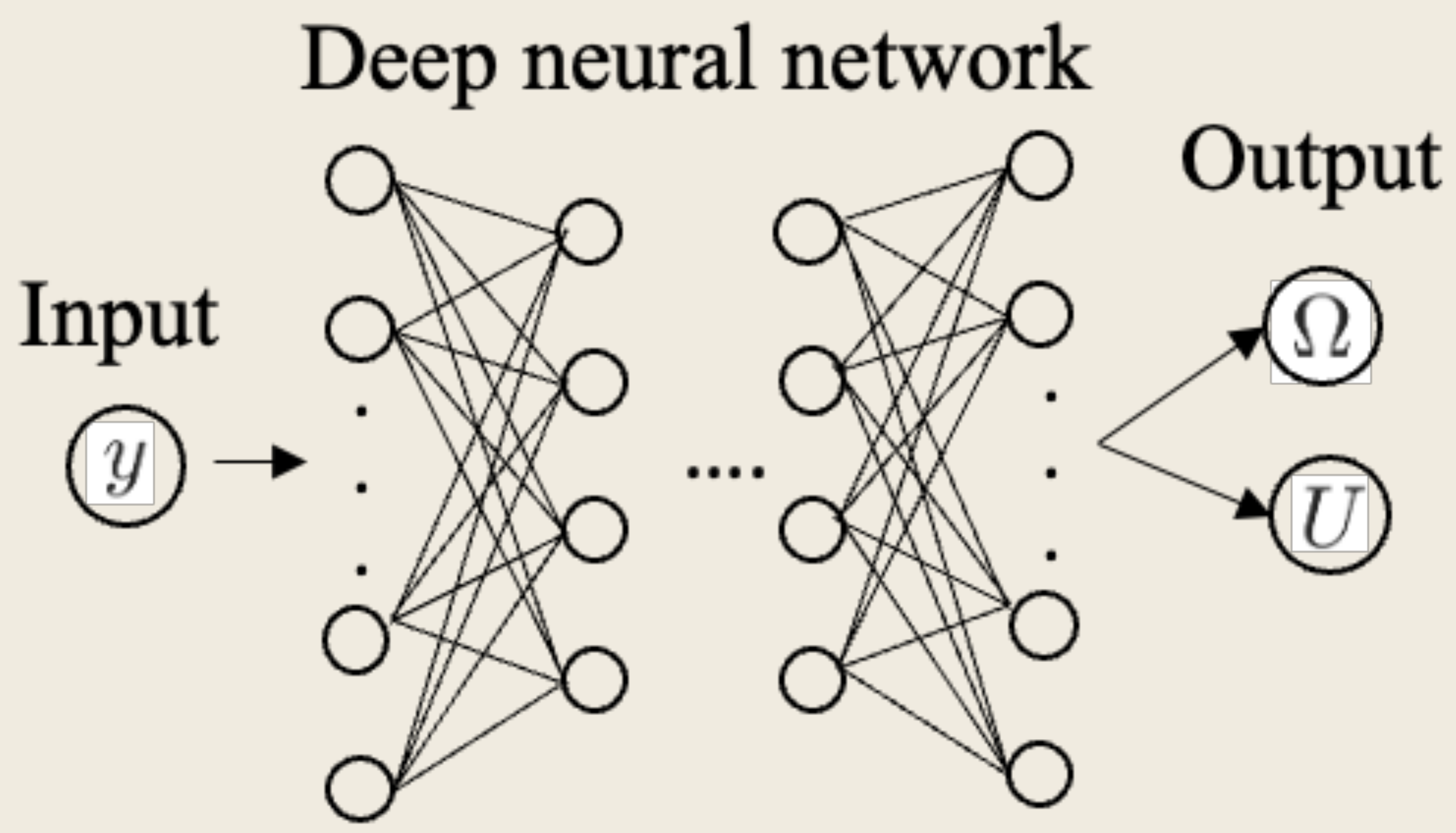
- Blow up occurs for $0 \leq \alpha < \frac{1}{2}$.
- Global-wellposedness holds for $\alpha \geq 1$.

In self-similar coordinates, the dissipative term decays exponentially if $\alpha(1+\lambda) < 1$. Thus, blow up for $0 \leq \alpha < 1$ is attainable using inviscid self-similar solutions if

$$\lambda < \frac{1}{\alpha} - 1.$$

To address the open range $\frac{1}{2} < \alpha < 1$, one needs $\lambda < 1$.

PINN setup for CCF equation



Equation residues:

$$f_1 = \Omega + ((1+\lambda)y - U)\partial_y\Omega - \Omega\partial_y U$$

$$f_2 = \partial_y U - \tilde{H}\Omega$$

where $\tilde{H}$ is a numerical Hilbert transform.

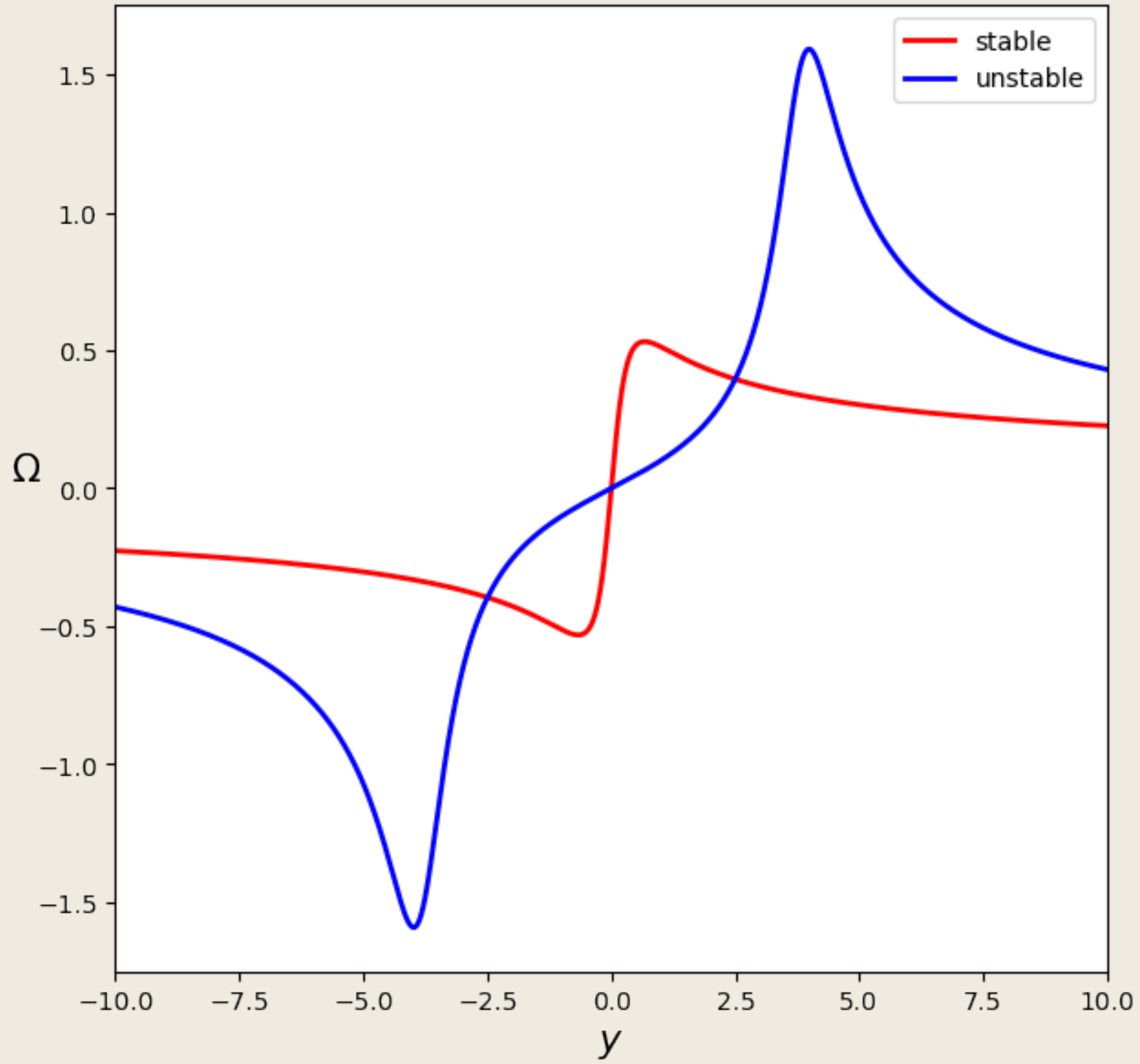
Additional constraints:

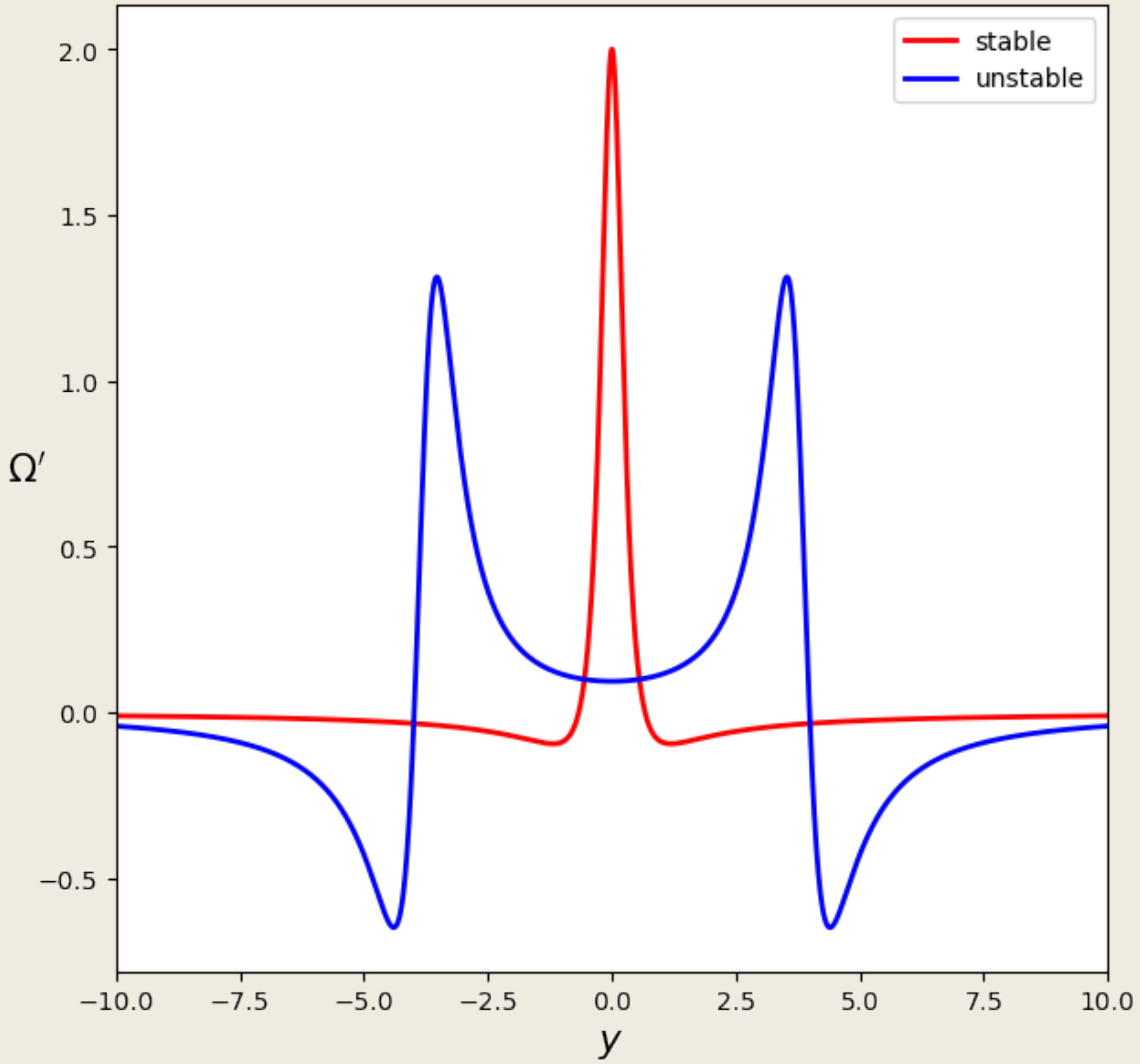
$\Omega(y_0) = c_0$, $\lim_{n\to\pm\infty} \Omega = 0$, Ω and U are both odd.

Denote the two networks as $\Omega(y, \mathbf{w}, \mathbf{b})$ and $U(y, \mathbf{w}, \mathbf{b})$, where $\mathbf{w}, \mathbf{b}$ are the weights and biases respectively, defining the network. As functions of y, Ω and U are smooth functions with explicit expressions, that can analytically differentiated.

The network is trained in terms of a loss function that samples the equation residues (and its derivatives) at random collocation points in the domain and takes into account the constraints.

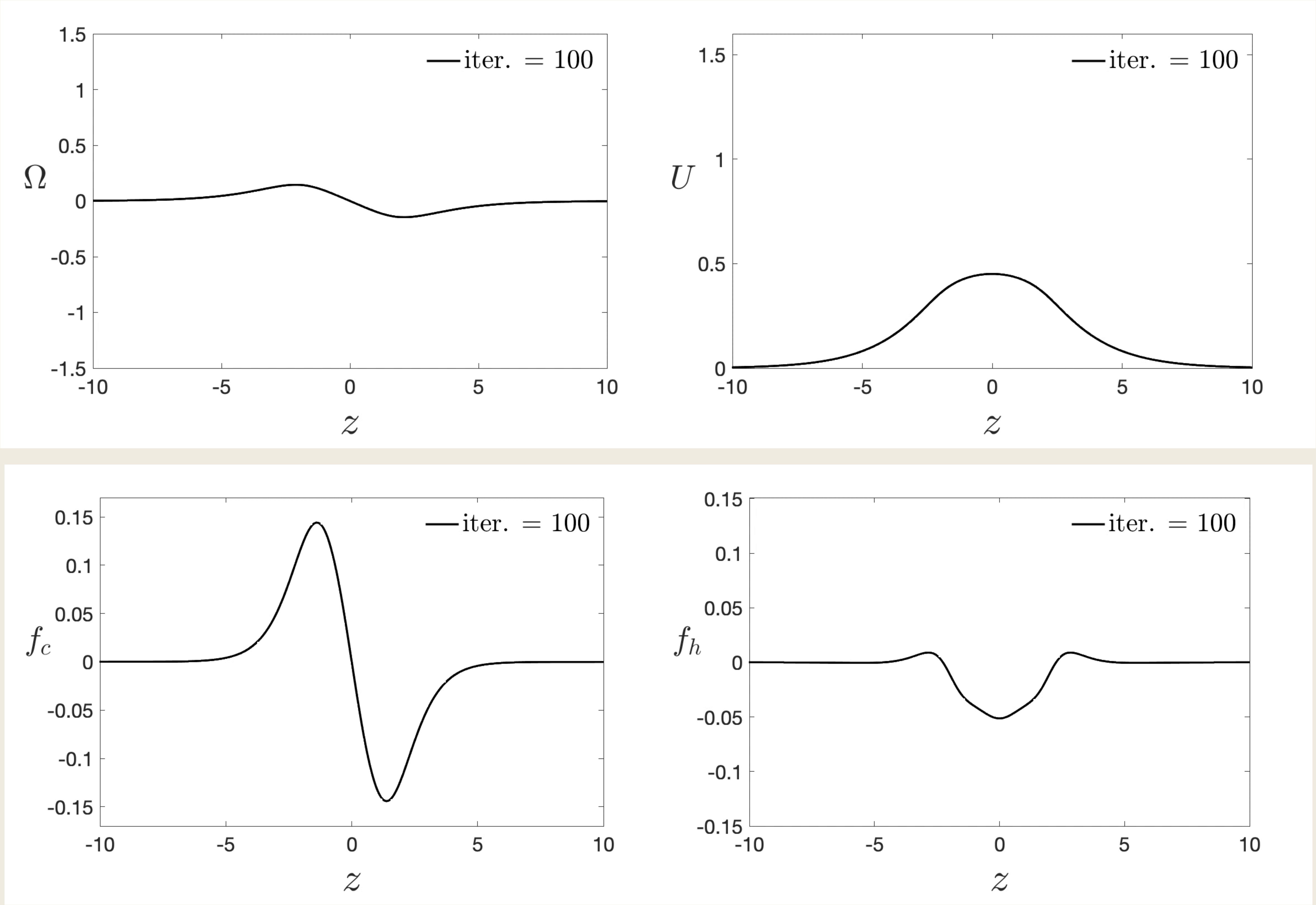
Results





The linearization of unstable solution has an extra unstable eigenvalue ≈ 0.367, in addition to the two eigenvalues $0,1$ from symmetries. For the stable solution $\lambda \approx 1.181$ and for the unstable solution $\lambda \approx 0.606$.

Early stage of training: Range: 100 - 50000. Frame gap is 100 iteration



Questions?