


Lectures on L-theory

Münster, Sept. 2020



Setup: R ring, M inv. module with involution over R , i.e.

- M is an $R \otimes R$ module, $\tau: R \otimes R \rightarrow R \otimes R$ flip isom.
- $\sigma: M \xrightarrow{\cong} \tau^*(M)$, $\sigma^2 = \text{id}$ i.e. $\sigma(r \otimes s \cdot x) = (s \otimes r) \cdot \sigma(x)$
- as R -module, M is f.g. projective $\neq R \xrightarrow{\cong} \text{End}_R(M)$

Examples: 1) R commutative, M invertible object in (Mod_R, \otimes)

i.e. a line bundle (rank 1 f.g. projective)

- $R \otimes R$ module str. through the map $R \otimes R \xrightarrow{m} R$
- any $\sigma \in \text{End}_R(M)$ of order 2 e.g. $\pm \text{id}_M$

2) R ring with (anti)involution: $i: R^{\text{op}} \xrightarrow{\cong} R$, $i^2 = \text{id}$, $\varepsilon \in R^\times$ central with $i(\varepsilon) \cdot \varepsilon = 1$
 set $M = R$ viewed as $R \otimes R \cong R \otimes R^{\text{op}}$ module

$\sigma(r) = \varepsilon \cdot i(r)$ e.g. $\varepsilon = \pm 1$

e.g. $R = (\mathbb{Z}G, i \text{ determined by } g \mapsto w_\pm(g) \cdot g^{-1})$
 for $w_\pm: G \rightarrow \{\pm 1\}$ orientation char.

Definition: Let R be a ring, M inv. mod. w/ inv. over R ,

$\text{Proj}(R) = \text{fin. gen. proj. } R\text{-modules}$

$D_M: \text{Proj}(R)^{\text{op}} \xrightarrow{\cong} \text{Proj}(R) : P \mapsto \text{Hom}_R(P, M)$ is the duality an. to M .

$b: P \otimes P \rightarrow M$

$\text{Hom}_{R \otimes R}(P \otimes P, M) = \text{bilinear } M\text{-valued forms on } P$

$\circlearrowleft C_2$: conjugate the flip action on $P \otimes P$ with σ

$\text{Hom}_{R \otimes R}(P \otimes P, M)_{C_2} = M\text{-valued quadr. forms}$

$\downarrow \text{sym.}$ $\text{Im}(\text{sym}) = M\text{-valued even forms}$

$\text{Hom}_{R \otimes R}(P \otimes P, M)^{C_2} = M\text{-valued sym. forms}$

$(b: P \otimes P \rightarrow M) \cong (\tilde{b}: P \rightarrow \text{Hom}_R(P, M))$

An M -valued sym. form is unimodular if the ass. map $P \xrightarrow{\cong} D_M P$ is an isom.

$P \in \text{Proj}(R) \rightsquigarrow \text{hyp}(P) = (P \otimes D_M P, \text{ev})$ is a sym. M -valued form

A sym. form b on P is called metabolic if \exists a Lagrangian, i.e.

Exercise 2: Show that $\text{hyp}(P)$ is canonically a quadratic form.

$L \subseteq P$ f.g. proj. sub module s.th. the sequence

$0 \rightarrow L \rightarrow P \xrightarrow{\tilde{b}} D_M P \rightarrow D_M L \rightarrow 0$

$b|_L = 0$.

is exact.

if (b, q) is quadr. and L a Lagrangian for b , we say that L is a quadr. Lagrangian

if in addition $q|_L = 0$.

Exercise 3: let (b, q) be a quadr. form on P
 Show: if \exists a quadr. Lagrangian L , then $(b, q) \cong \text{hyp}(L)$. Hint: wait till end of 2nd lecture

• Show that not every sym. metabolic form is hyperbolic

Exercise 4: Show that $W^s(R, M)$ and $W^q(R, M)$

are abelian groups and show that the sym. induces a group homom. $W^q(R, M) \rightarrow W^s(R, M)$.

Hint: Show that $-[P, b] = [P, -b]$

Definition: R, M as before.

$$W^s(R; M) = \left\{ \begin{array}{l} \text{isom. classes of unimod.} \\ \text{M-valued sym. forms} \end{array} \right\} / \langle \text{metabolic forms} \rangle$$

$$W^q(R; M) = \left\{ \begin{array}{l} \text{isom. classes of unimod.} \\ \text{M-valued quadr. forms} \end{array} \right\} / \langle \text{quadr. metabolic forms} \rangle$$

Notation: R comm, $M = R$, $\left. \begin{array}{l} \sigma = \text{id} \rightsquigarrow W^s(R; M) = W^s(R) \\ \sigma = -\text{id} \rightsquigarrow W^s(R; M) = W^s(R) \end{array} \right\}$ $W^q(R; M) = W^q(R)$
 $W^q(R; M) = W^q(R)$

Prop: K a field. Then $W^s(K)$ is generated by forms $\langle u \rangle = (K, x, y \mapsto u \cdot xy)$ for $u \in K^\times$

proof: Let (V, b) be a sym. bilinear form

$$\{y \in V \mid b(x, y) = 0\} = \ker(b(x, -): V \rightarrow K)$$

Exercise 5: K field, $\text{char}(K) \neq 2$

Show $W^s(K) = 0$.

eg. quadr. closed fields.

Assume $\exists x \in V$ s.t. $b(x, x) = u \neq 0$. Let $V' = \langle x \rangle^\perp$, $b' = b|_{V'}$

$\bullet K$ s.t. $K^\times \xrightarrow{f^2} K^\times$ surjective $\rightarrow W^s(K) \cong \mathbb{Z}/2$

Check: $(V, b) \cong \langle u \rangle \oplus (V', b')$. Inductively we get

$$(V, b) \cong \langle u_1 \rangle \oplus \dots \oplus \langle u_k \rangle \oplus (W, b')$$

Exercise 6: Suppose $\text{char}(K) \neq 2$, (W, b') s.t. $b'(x, x) = 0 \forall x \in W$. Show $W = 0$.

To finish the proof, it suffices to show that $[W, b'] = 0$ in $W^s(K)$.

Fix $x \in W$. Let y be s.t. $b'(x, y) = 1$. Let $\bar{W} = \langle x, y \rangle^\perp$, $\bar{b}' = b'|_{\bar{W}}$.

Then $\langle x \rangle \subseteq \langle x, y \rangle$ is a Lagrangian. $\Rightarrow [W, b'] = [\bar{W}, \bar{b}'] \in W^s(K)$. Now induct \square

Definition: A Poincaré co-cat. is a tuple $(\mathcal{C}, \mathcal{Q})$ with

$\bullet \mathcal{C}$ small Abelian ∞ -category

ring / rng spectrum \swarrow \searrow \mathcal{C} is scheme
 eg. $\mathcal{C} = \text{Sp}^{\text{fin}}$, $\mathcal{D}^p(R)$, $\mathcal{D}^p(X)$

$\bullet \mathcal{Q}: \mathcal{C}^{\text{op}} \rightarrow \text{Sp}$ reduced, 2-excisive $(\mathcal{Q}(0) = 0)$

in part. $B_{\mathcal{Q}}(x, y) = \text{cobb}(\mathcal{Q}(x) \otimes \mathcal{Q}(y) \rightarrow \mathcal{Q}(x \otimes y))$ is symmetric bilinear

s.t. $B_{\mathcal{Q}}(x, y) = \text{map}_{\mathcal{C}}(X, \mathcal{D}_{\mathcal{Q}} Y)$ for some duality equivalence $\mathcal{D}_{\mathcal{Q}}: \mathcal{C}^{\text{op}} \xrightarrow{\cong} \mathcal{C}$

$(\mathcal{C}, \mathcal{D}_{\mathcal{Q}}) \in \text{Cat}_{\infty}^{\text{ex}}$

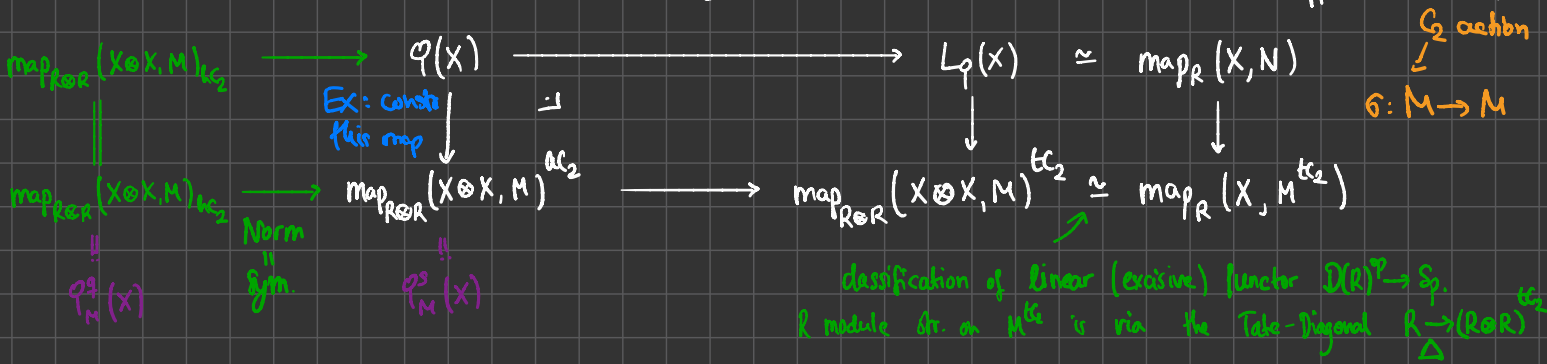
Definition: R, M as in the beginning. A Poincaré structure

$\mathcal{Q}: \mathcal{D}^p(R) \rightarrow \text{Sp}$ is called compatible with M

this means that the coass effect $B_{\mathcal{Q}}$ of \mathcal{Q} is given by $(X, Y) \mapsto \text{map}_{\text{Mod } R} (X \otimes Y, M)$.

if $\mathcal{D}_{\mathcal{Q}} = \mathcal{D}_M = \text{map}_R(-, M): \mathcal{D}^p(R)^{\text{op}} \xrightarrow{\cong} \mathcal{D}^p(R)$.

Fact: reduced, 2 excisive functors are classified by their cross-effect B_q and their excisive approximation L_q



Consequently, for R, M as always, compatible Poincaré structures on $\mathcal{D}^p(R)$ are described by

- an R -module N , and
- an R -linear map $N \xrightarrow{\alpha} M^{C_2}$.

- e.g.:
- $N=0 \Rightarrow \mathcal{P}_M^q$
 - $N=M^{C_2}, \alpha=id \Rightarrow \mathcal{P}_M^s$
 - $N=\tau_{\mathbb{Z}_2} M^{C_2}, \alpha \text{ can.} \Rightarrow \mathcal{P}_M^{\geq m}$

Exercise 7: Show that for $P \in \text{Proj}(R)$ we have

- $\mathcal{P}_M^{\geq 0}(P) = \text{Hom}_{R\otimes R}(P\otimes P, M)^{C_2}$
 - $\mathcal{P}_M^{\geq 2}(P) = \text{Hom}_{R\otimes R}(P\otimes P, M)_{C_2}$
 - $\mathcal{P}_{\mathbb{Z}}^b(\mathbb{Z}) = \text{Burnside ring of } C_2$.
- Equivalences: $\mathcal{P}_M^{\geq 0} = \mathcal{P}_M^s$, $\mathcal{P}_M^{\geq 2} = \mathcal{P}_M^{\geq m}$

- R comm, $M=R, N=R, \alpha = \text{Frobenius}$
 $R \rightarrow (R\otimes R)^{C_2} \xrightarrow{m} R^{C_2} \Rightarrow \mathcal{P}_R^{\text{off}}$
- R comm, $M=R, N=\tau_{\mathbb{Z}_2} R^{C_2} = \tau_{\mathbb{Z}_2} R^{C_2} \times_{R/2} R$
 $\Rightarrow \mathcal{P}_R^b$

Definition: the space $\text{Fm}(\mathcal{E}, \mathcal{Q})$ of forms on $(\mathcal{E}, \mathcal{Q})$ is the groupoid core of $\int_{\mathcal{E}^{\mathcal{Q}}} \Omega^{\text{univ}} \mathcal{Q}: \mathcal{E}^{\mathcal{Q}} \rightarrow \text{Spec}$

i.e. the collection of pairs (X, q) with $X \in \mathcal{E}, q \in \Omega^{\text{univ}} \mathcal{Q}(X)$.

such a pair is called a Poincaré object if $q_{\#}: X \xrightarrow{\cong} D_q X$ is an equivalence.

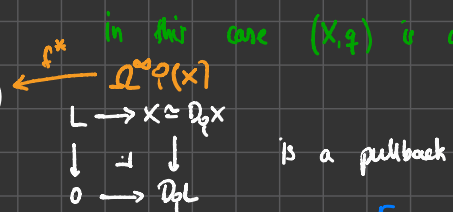
Let $\text{Pn}(\mathcal{E}, \mathcal{Q}) \subseteq \text{Fm}(\mathcal{E}, \mathcal{Q})$ be all components whose objects are Poincaré.

Definition: A Lagrangian of a Poincaré object (X, q) consists of a pair (L, γ) where

- $L \xrightarrow{f} X$ is a map and

- $\gamma: f^*(q) \sim 0$ in $\Omega^{\text{univ}} \mathcal{Q}(L)$

such that the induced diagram



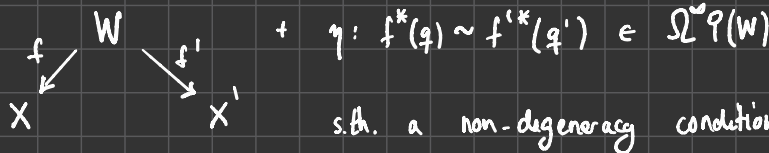
Definition: $L_0(\mathcal{E}, \mathcal{Q}) = \left\{ \begin{array}{l} \text{equiv. classes of} \\ \text{Poincaré objects} \\ \text{in } (\mathcal{E}, \mathcal{Q}) \end{array} \right\} \oplus \left\{ \begin{array}{l} \text{metabolic objects} \end{array} \right\}$

Exercise 8: 1) $L_0(\mathcal{E}, \mathcal{Q})$ is a group
 2) the relation $(X, q) \sim (X', q')$
 $(X \oplus X', q \oplus q')$ is metabolic is a congruence relation (i.e. an equiv. relation compatible w/ \oplus).

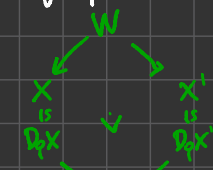
3) $L_0(\mathcal{E}, \mathcal{Q}) = L$ -group defined by Yokota.

Next goal: Define an L-theory space $L(\mathcal{C}, \mathcal{Q})$ and describe the higher L-groups

Observation: $(X, q) \sim (X', q') \iff \exists$ diagram of the shape



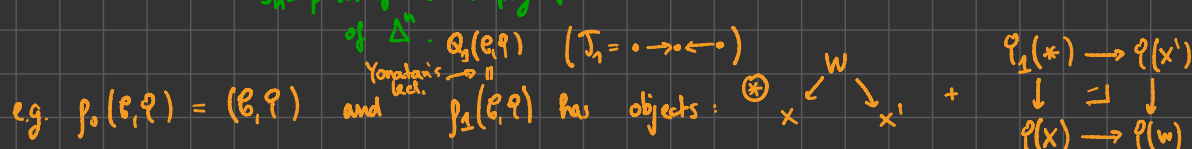
s.t. a non-degeneracy condition is satisfied. the square should be a pullback.



Definition: let $(\mathcal{C}, \mathcal{Q})$ be a Poincaré ∞ -category. Define

$$\text{Cat}_{\infty}^{\mathcal{P}} \ni \mathcal{P}_n(\mathcal{C}, \mathcal{Q}) = \left[\text{Fun}(J_n^{\text{op}}, \mathcal{C}), \text{Fun}(J_n^{\text{op}}, \mathcal{C})^{\text{op}} \simeq \text{Fun}(J_n, \mathcal{C}^{\text{op}}) \xrightarrow{p_n} \text{Fun}(J_n, \mathcal{S}p) \xrightarrow{\text{lim}} \mathcal{S}p \right]$$

$J_n =$ poset of non-empty faces of Δ^n .



Fact: Poincaré objects of $\mathcal{P}_1(\mathcal{C}, \mathcal{Q}) \hat{=} \text{Lagrangean in } (X \otimes X', q \otimes q') = \text{cobordisms between } (X, q) \text{ and } (X', q')$

Proposition: there is a functor $\Delta^{\text{op}} \times \text{Cat}_{\infty}^{\mathcal{P}} \rightarrow \text{Cat}_{\infty}^{\mathcal{P}}$ sending $([n], (\mathcal{C}, \mathcal{Q})) \mapsto \mathcal{P}_n(\mathcal{C}, \mathcal{Q})$.

In addition, all simplicial face maps $\mathcal{P}_n(\mathcal{C}, \mathcal{Q}) \rightarrow \mathcal{P}_{n-1}(\mathcal{C}, \mathcal{Q})$ are split Poincaré-Vietoris projections.

Definition: $L: \text{Cat}_{\infty}^{\mathcal{P}} \rightarrow \text{Spc}$ is the geometric realisation of

$$[n] \mapsto \text{Pn}(\mathcal{P}_n(\mathcal{C}, \mathcal{Q}))$$

$\text{Pn}(-) = \text{Map}_{\text{Cat}_{\infty}^{\mathcal{P}}}(\mathcal{P}_n^{\text{th}}, \mathcal{P}^{\text{th}}) \rightarrow \text{Cat}_{\infty}^{\mathcal{P}}$ pre-additive.

$\text{Pn}(\mathcal{C}, \mathcal{Q})$ is an E_{∞} -space for all $(\mathcal{C}, \mathcal{Q}) \rightsquigarrow L(\mathcal{C}, \mathcal{Q})$ is also an E_{∞} -space

Exercise 9: Show that $\pi_0 L(\mathcal{C}, \mathcal{Q}) = L_0(\mathcal{C}, \mathcal{Q})$.

Deduce that $L(\mathcal{C}, \mathcal{Q})$ is a grouplike E_{∞} -space and hence a connective spectrum

Exercise 10: Show that $|[n] \mapsto \text{Fm}(\mathcal{P}_n(\mathcal{C}, \mathcal{Q}))|$ is contractible.

Hint: use extension by 0's to define an extra degeneracy.