Lectures on L-theory

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Setup: R ring, M inv. module with involution over R, i.e.

- M is an $R \otimes_{\mathbb{Z}} R$ module, $\tau: R \otimes_{\mathbb{Z}} R \to R \otimes_{\mathbb{Z}} R$ flip isom.
- $\sigma: M \xrightarrow{\cong} \tau^*(M)$ $\sigma^2 = \mathrm{id}$ i.e. $\sigma((r \otimes s) \cdot x) = (s \otimes r) \cdot \sigma(x)$
- as R-module, M is f.g. projective & $R \xrightarrow{\cong} \mathrm{End}_R(M)$

Examples: 1) R commutative, M invertible object in $(\mathrm{Mod}_R, \otimes_R)$ i.e. a line bundle (rank 1 f.g. projective)
- $R \otimes R$ module str. through the map $R \otimes R \xrightarrow{m} R$
- any $\sigma \in \mathrm{End}_R(M)$ of order 2 e.g. $\pm \mathrm{id}_M$

2) R ring with (anti) involution: $i: R^{op} \xrightarrow{\cong} R$, $i^2 = \mathrm{id}$, $\varepsilon \in R^\times$ central with $i(\varepsilon) \cdot \varepsilon = 1$

set
- $M := R$ viewed as $R \otimes R \overset{i}{\cong} R \otimes R^{op}$ module
- $\sigma(r) := \varepsilon \cdot i(r)$ e.g. $\varepsilon = \pm 1$

e.g. $R = (\mathbb{Z}G, i$ determined by $g \mapsto w_1(g) \cdot g^{-1})$ for $w_1: G \to \{\pm 1\}$ orientation char.

Definition: Let R be a ring, M inv. mod. w/ inv. over R, $\mathrm{Proj}(R)$ = fin. gen. proj. R-modules

- $D_M: \mathrm{Proj}(R)^{op} \xrightarrow{\cong} \mathrm{Proj}(R)$: $P \longmapsto \mathrm{Hom}_R(P, M)$ is the duality an. to M.

$b: P \otimes P \to M$
- $\mathrm{Hom}_{R \otimes R}(P \otimes P, M)$ = bilinear M-valued forms on P

$\circlearrowleft C_2 = \{1, T\}$: conjugate the flip action on $P \otimes P$ with σ

- $\mathrm{Hom}_{R \otimes R}(P \otimes P, M)_{C_2}$ = M-valued quadr. forms

$x \mapsto x + T(x)$ $\downarrow$ sym. $\mathrm{Im}(\mathrm{sym})$ = M-valued even forms

- $\mathrm{Hom}_{R \otimes R}(P \otimes P, M)^{C_2}$ = M-valued sym. forms

Exercise 1: Show that M-valued quadr. forms are equivalently described by: a) $b: P \otimes P \to M$ sym. bilinear

b) $q: P \to M_{C_2}$ s.th.
- $q(rx) = (r \otimes r) \cdot [q(x)]$
- $q(x+y) - q(x) - q(y) = [b(x,y)]$
- $b(x,x) = q(x) + \sigma(q(x))$

$(b: P \otimes P \to M) \hat{=} (\tilde{b}: P \to \mathrm{Hom}_R(P, M))$

- An M-valued sym. form is unimodular if the ass. map $P \xrightarrow{\cong} D_M P$ is an isom.

- $P \in \mathrm{Proj}(R) \rightsquigarrow \mathrm{hyp}(P) = (P \oplus D_M P, \mathrm{ev})$ is a sym. M-valued form unimod.

Exercise 2: Show that $\mathrm{hyp}(P)$ is canonically a quadratic form.

- A sym. form b on P is called metabolic if $\exists$ a Lagrangian, i.e

$L \subseteq P$ f.g. proj. sub module s.th. the sequence $b|_L = 0$.

$$0 \to L \to P \overset{\tilde{b}}{\cong} D_M P \longrightarrow D_M L \longrightarrow 0$$ is exact.

if (b,q) is quadr. and L a Lagrangian for b, we say that L is a quadr. Lagrangian if in addition $q|_L = 0$.

Exercise 3:
- let (b,q) be a quadr. form on P. Show: if $\exists$ a quadr. Lagrangian L, then $(b,q) \cong \mathrm{hyp}(L)$. Hint: wait till end of 2^{nd} lecture
- Show that not every metabolic sym. form is hyperbolic

Exercise 4: Show that $W^s(R,M)$ and $W^q(R,M)$ are abelian groups and show that the sym. induces a group homom. $W^q(R;M) \longrightarrow W^s(R;M)$.

Hint: Show that $-[P,b] = [P,-b]$

Definition: R, M as before.

- $W^s(R;M) = \left\{ \begin{matrix} \text{isom. classes of unimod.} \\ M\text{-valued sym. forms} \end{matrix}, \oplus \right\} \Big/ \langle \text{metabolic forms} \rangle$
- $W^q(R;M) = \left\{ \begin{matrix} \text{isom. classes of unimod.} \\ M\text{-valued quadr. forms} \end{matrix}, \oplus \right\} \Big/ \langle \text{quadr. metabolic forms} \rangle$

Notation: R comm, $M = R$, $\begin{cases} \sigma = \mathrm{id} \rightsquigarrow W^s(R;M) = W^s(R) & W^q(R;M) = W^q(R) \\ \sigma = -\mathrm{id} \rightsquigarrow W^s(R;M) = W^{-s}(R) & W^q(R;M) = W^{-q}(R) \end{cases}$

Prop: K a field. Then $W^s(K)$ is generated by forms $\langle u \rangle = (K, x \otimes y \mapsto u \cdot xy)$ for $u \in K^\times$

proof: Let (V,b) be a sym. bilinear form

$\{y \in V \mid b(x,y) = 0\} = \ker(b(x,-) : V \to K)$

Assume $\exists\, x \in V$ s.th. $b(x,x) = u \neq 0$. Let $V' = \langle x \rangle^\perp$, $b' = b|_{V'}$

Check: $(V,b) \cong \langle u \rangle \oplus (V', b')$. Inductively we get

$(V,b) \cong \langle u_1 \rangle \oplus \dots \oplus \langle u_k \rangle \oplus (W, b')$ and $b'(x,x) = 0 \ \forall\, x \in W$.

To finish the proof, it suffices to show that $[W, b'] = 0$ in $W^s(K)$.

Fix $x \in W \setminus \{0\}$. Let y be s.th. $b'(x,y) = 1$. Let $\overline{W} = \langle x, y \rangle^\perp$, $\overline{b'} = b'|_{\overline{W}}$.

Then $\langle x \rangle \subseteq \langle x, y \rangle$ is a Lagrangian. $\Rightarrow [W, b'] = [\overline{W}, \overline{b'}] \in W^s(K)$. Now induct $\blacksquare$

Exercise 5: K field, $\mathrm{char}(K) \neq 2$

- Show $W^s(K) = 0$. (e.g. quadr. closed fields.)
- K s.th. $K^\times \xrightarrow{(-)^2} K^\times$ surjective $\Rightarrow W^s(K) \cong \mathbb{Z}/2$

Exercise 6: Suppose $\mathrm{char}(K) \neq 2$, (W, b') s.th. $b'(x,x) = 0 \ \forall\, x \in W$. Show $W = 0$.

Definition: A Poincaré ∞-cat. is a tuple $(\mathcal{C}, \varphi)$ with

- $\mathcal{C}$ small stable ∞-category   e.g. $\mathcal{C} = \mathrm{Sp}^{fin}$, $\mathcal{D}^p(R)$, $\mathcal{D}^p(X)$, (ring / ring spectrum; qcqs scheme)
- $\varphi : \mathcal{C}^{op} \to \mathrm{Sp}$ reduced, 2-excisive ($\varphi(0) = 0$)   in part. $B_\varphi(X,Y) = \mathrm{cofib}(\varphi(X) \oplus \varphi(Y) \to \varphi(X \oplus Y))$ is symmetric bilinear

s.th. $B_\varphi(X,Y) = \mathrm{map}_{\mathcal{C}}(X, D_\varphi Y)$ for some duality equivalence $D_\varphi : \mathcal{C}^{op} \xrightarrow{\simeq} \mathcal{C}$ $(\mathcal{C}, D_\varphi) \in (\mathrm{Cat}_\infty^{ex})^{hC_2}$

Definition: R, M as in the beginning. A Poincaré structure

$\varphi : \mathcal{D}^p(R) \longrightarrow \mathrm{Sp}$ (perfect complexes on R) is called compatible with M

if $D_\varphi = D_M = \mathrm{map}_R(-, M) : \mathcal{D}^p(R)^{op} \xrightarrow{\simeq} \mathcal{D}^p(R)$.

This means that the cross effect B_φ of φ is given by $(X,Y) \mapsto \mathrm{map}_{R \otimes R}(X \otimes Y, M)$.

Fact: reduced, 2 excisive functors are classified by their cross-effect B_φ and their excisive approximation L_φ

C_2 action $\sigma : M \to M$

$$\begin{tikzcd}
\mathrm{map}_{R\otimes R}(X\otimes X, M)_{hC_2} \arrow[r] \arrow[d, equal] & \varphi(X) \arrow[r] \arrow[d, "\text{Ex: constr this map}"'] \arrow[dr, phantom, "\lrcorner"] & L_\varphi(X) \arrow[r, phantom, "\simeq"] \arrow[d] & \mathrm{map}_R(X, N) \arrow[d] \\
\mathrm{map}_{R\otimes R}(X\otimes X, M)_{hC_2} \arrow[r, "\text{Norm}"'] & \mathrm{map}_{R\otimes R}(X\otimes X, M)^{hC_2} \arrow[r] & \mathrm{map}_{R\otimes R}(X\otimes X, M)^{tC_2} \arrow[r, phantom, "\simeq"] & \mathrm{map}_R(X, M^{tC_2})
\end{tikzcd}$$

$\mathrm{map}_{R\otimes R}(X\otimes X, M)_{hC_2} = \varphi^q_M(X)$, Norm = Sym., $\mathrm{map}_{R\otimes R}(X\otimes X, M)^{hC_2} = \varphi^s_M(X)$

classification of linear (excisive) functor $D(R)^{op} \to Sp$.
R module str. on M^{tC_2} is via the Tate-Diagonal $R \xrightarrow{\Delta} (R\otimes R)^{tC_2}$

Consequently, for R, M as always, compatible Poincaré structures on $\mathcal{D}^p(R)$ are described by

- an R-module N, and
- an R-linear map $N \xrightarrow{\alpha} M^{tC_2}$.

e.g.
- $N = 0 \Rightarrow \varphi^q_M$
- $N = M^{tC_2}$, $\alpha = id \Rightarrow \varphi^s_M$
- $N = \tau_{\geq m} M^{tC_2}$, α can. $\Rightarrow \varphi^{\geq m}_M$
- R comm, $M = R$ ($\sigma = id$), $N = R$, α = Frobenius $R \to (R\otimes R)^{tC_2} \xrightarrow{m} R^{tC_2} \Rightarrow \varphi^{TF}_R$
- R comm. $M = R$ ($\sigma = id$), $N = \tau_{\geq 1/2} R^{tC_2} = \tau_{\geq 0} R^{tC_2} \times_{R/2} R \Rightarrow \varphi^b_R$

Exercise 7: Show that for $P \in \mathrm{Proj}(R)$ we have

- $\varphi^{\geq 0}_M(P) = \mathrm{Hom}_{R\otimes R}(P\otimes P, M)^{C_2}$
- $\varphi^{\geq 1}_M(P) = \mathrm{Hom}_{R\otimes R}(P\otimes P, M)_{C_2}$

$\Leftrightarrow$ $\varphi^{\geq 0}_M = \varphi^{gs}_M$, $\varphi^{\geq 1}_M = \varphi^{gq}_M$

- $\varphi^b_{\mathbb{Z}}(\mathbb{Z})$ = Burnside ring of C_2.

Definition: the space $\mathrm{Fm}(\mathcal{C}, \varphi)$ of forms on $(\mathcal{C}, \varphi)$ is the groupoid core of $\int_{\mathcal{C}^{op}} \Omega^\infty \varphi : \mathcal{C}^{op} \to Spc$

i.e. the collection of pairs (X, q) with $X \in \mathcal{C}$, $q \in \Omega^\infty \varphi(X)$.

such a pair is called a Poincaré object if $q_\# : X \xrightarrow{\simeq} D_\varphi X$ is an equivalence. Ex': constr this map.

Let $\mathrm{Pn}(\mathcal{C}, \varphi) \subseteq \mathrm{Fm}(\mathcal{C}, \varphi)$ be all components whose objects are Poincaré.

Definition: A Lagrangian of a Poincaré object (X, q) consists of a pair (L, η) where

- $L \xrightarrow{f} X$ is a map and
- $\eta : f^*(q) \sim 0$ in $\Omega^\infty \varphi(L) \xleftarrow{f^*} \Omega^\infty \varphi(X)$

in this case (X, q) is called metabolic.

such that the induced diagram

$$\begin{tikzcd}
L \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \simeq D_\varphi X \arrow[d] \\
0 \arrow[r] & D_\varphi L
\end{tikzcd}$$

is a pullback.

Definition: $L_0(\mathcal{C}, \varphi) = \left\{ \begin{matrix} \text{equiv. classes of} \\ \text{Poincaré objects} \\ \text{in } (\mathcal{C}, \varphi) \end{matrix}, \oplus \right\} / \langle \text{metabolic objects} \rangle$

Exercise 8:
1) $L_0(\mathcal{C}, \varphi)$ is a group.
2) the relation $(X, q) \sim (X', q') \overset{Def}{\Leftrightarrow} (X \oplus X', q \oplus -q')$ is metabolic is a congruence relation (i.e. an equiv. relation compatible w/ $\oplus$).
3) $L_0(\mathcal{C}, \varphi)$ = L-group defined by Yonatan.

Next goal: Define an L-theory space $\mathcal{L}(\mathcal{C},Ϙ)$ and describe the higher L-groups

Observation: $(X,q) \sim (X',q') \Leftrightarrow \exists$ diagram of the shape

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ X & & X' \end{tikzcd} \quad + \quad \eta : f^*(q) \sim f'^*(q') \in \Omega^\infty Ϙ(W)$$

s.th. a non-degeneracy condition is satisfied: the square should be a pullback.

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner"] & \\ X \arrow[d, phantom, "\wr"] & & X' \arrow[d, phantom, "\wr"] \\ D_Ϙ X \arrow[dr] & & D_Ϙ X' \arrow[dl] \\ & D_Ϙ W & \end{tikzcd}$$

Definition: Let $(\mathcal{C},Ϙ)$ be a Poincaré ∞-category. Define

$$\mathrm{Cat}^p_\infty \ni \mathcal{Q}_n(\mathcal{C},Ϙ) = \left[\mathrm{Fun}(T_n^{op}, \mathcal{C}),\ \mathrm{Fun}(T_n^{op},\mathcal{C})^{op} \simeq \mathrm{Fun}(T_n, \mathcal{C}^{op}) \xrightarrow{Ϙ_*} \mathrm{Fun}(T_n, \mathrm{Sp}) \xrightarrow{\lim} \mathrm{Sp} \right]$$

T_n = poset of non-empty faces of Δ^n.

$\mathcal{Q}_1(\mathcal{C},Ϙ)$ $\quad (T_1 = \bullet \to \bullet \leftarrow \bullet)$

Yoneda's lem.

e.g. $\mathcal{Q}_0(\mathcal{C},Ϙ) = (\mathcal{C},Ϙ)$ and $\mathcal{Q}_1(\mathcal{C},Ϙ)$ has objects: (*) $X \leftarrow W \rightarrow X'$ +

$$\begin{tikzcd} Ϙ_1(*) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Ϙ(X') \arrow[d] \\ Ϙ(X) \arrow[r] & Ϙ(W) \end{tikzcd}$$

Fact: Poincaré objects of $\mathcal{Q}_1(\mathcal{C},Ϙ)$ $\hat{=}$ Lagrangian in $(X \oplus X', q \oplus -q')$ $=:$ cobordisms between (X,q) and (X',q').

Proposition: there is a functor $\Delta^{op} \times \mathrm{Cat}^p_\infty \longrightarrow \mathrm{Cat}^p_\infty$ sending $([n],(\mathcal{C},Ϙ))$ to $\mathcal{Q}_n(\mathcal{C},Ϙ)$.

In addition, all simpl. face maps $\mathcal{Q}_n(\mathcal{C},Ϙ) \xrightarrow{d_i} \mathcal{Q}_{n-1}(\mathcal{C},Ϙ)$ are split Poincaré-Verdier projections.

Definition: $\mathcal{L} : \mathrm{Cat}^p_\infty \to \mathrm{Spc}$ is the geometric realisation of

$$[n] \longmapsto \mathrm{Pn}(\mathcal{Q}_n(\mathcal{C},Ϙ))$$

$\mathrm{Pn}(-) = \mathrm{Map}_{\mathrm{Cat}^p_\infty}((\mathrm{Sp}^{fin}, Ϙ^u), -)$ + Cat^p_∞ is pre additive.

$\Downarrow$

$\mathrm{Pn}(\mathcal{C},Ϙ)$ is an $\mathbb{E}_\infty$-space for all $(\mathcal{C},Ϙ)$ $\rightsquigarrow$ $\mathcal{L}(\mathcal{C},Ϙ)$ is also an $\mathbb{E}_\infty$-space

Exercise 9: Show that $\pi_0 \mathcal{L}(\mathcal{C},Ϙ) \cong L_0(\mathcal{C},Ϙ)$.
Deduce that $\mathcal{L}(\mathcal{C},Ϙ)$ is a grouplike $\mathbb{E}_\infty$-space and hence a connective spectrum

Exercise 10: Show that $\left| [n] \mapsto \mathrm{Fm}(\mathcal{Q}_n(\mathcal{C},Ϙ)) \right|$ is contractible.
Hint: use extension by 0's to define an extra degeneracy.