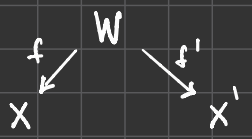


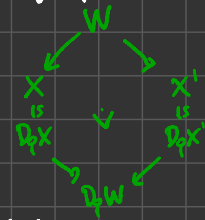
Next goal: Define an L-theory space $L(\mathcal{E}, \mathcal{Q})$ and describe the higher L-groups

Observation: $(X, q) \sim (X', q') \iff \exists$ diagram of the shape



$$+ \eta: f^*(q) \sim f'^*(q') \in \Omega^{\text{or}} \mathcal{Q}(W)$$

s.t. a non-degeneracy condition is satisfied



Definition: Let $(\mathcal{E}, \mathcal{Q})$ be a Poincaré ∞ -category. Define

$$\text{Cat}_{\infty}^{\mathcal{P}} \ni \mathcal{P}_n(\mathcal{E}, \mathcal{Q}) = \left[\text{Fun}(J_n^{\text{op}}, \mathcal{E}), \text{Fun}(J_n^{\text{op}}, \mathcal{E})^{\text{op}} \simeq \text{Fun}(J_n, \mathcal{E}^{\text{op}}) \xrightarrow{p_n} \text{Fun}(J_n, \mathcal{E}) \xrightarrow{\text{lim}} \text{Sp} \right]$$

$J_n =$ poset of non-empty faces of Δ^n

$$(\mathcal{E}, \mathcal{Q}) \text{ of } \Delta^n \text{ (Yonatan's lect. } \rightarrow \parallel \text{)}$$

e.g. $\mathcal{P}_0(\mathcal{E}, \mathcal{Q}) = (\mathcal{E}, \mathcal{Q})$ and $\mathcal{P}_1(\mathcal{E}, \mathcal{Q})$ has objects: $\begin{array}{ccc} & W & \\ \otimes \swarrow & & \searrow \\ X & & X' \end{array} + \begin{array}{ccc} \mathcal{Q}_1(*) & \rightarrow & \mathcal{Q}(X') \\ \downarrow \cong & & \downarrow \\ \mathcal{Q}(X) & \rightarrow & \mathcal{Q}(W) \end{array}$

Fact: Poincaré objects of $\mathcal{P}_1(\mathcal{E}, \mathcal{Q}) \hat{=} \text{lagrangian in } (X \boxtimes X', \mathcal{Q} \boxtimes \mathcal{Q}) = \text{cobordisms between } (X, q) \text{ and } (X', q')$

Proposition: there is a functor $\Delta^{\text{op}} \times \text{Cat}_{\infty}^{\mathcal{P}} \rightarrow \text{Cat}_{\infty}^{\mathcal{P}}$ sending $([n], (\mathcal{E}, \mathcal{Q})) \mapsto \mathcal{P}_n(\mathcal{E}, \mathcal{Q})$.

In addition, all simplicial face maps $\mathcal{P}_n(\mathcal{E}, \mathcal{Q}) \rightarrow \mathcal{P}_{n-1}(\mathcal{E}, \mathcal{Q})$ are split Poincaré-Vershik projections.

Definition: $L: \text{Cat}_{\infty}^{\mathcal{P}} \rightarrow \text{Spc}$ is the geometric realisation of

$$[n] \mapsto \mathcal{P}_n(\mathcal{E}, \mathcal{Q})$$

$\mathcal{P}_n(L) = \text{Map}_{\text{Cat}_{\infty}^{\mathcal{P}}}(\mathcal{P}_n^{\text{th}}, \mathcal{P}^{\text{th}}) \rightarrow \text{Cat}_{\infty}^{\mathcal{P}}$ is pre-additive.

$\mathcal{P}_n(\mathcal{E}, \mathcal{Q})$ is an E_n -space for all $(\mathcal{E}, \mathcal{Q}) \rightsquigarrow L(\mathcal{E}, \mathcal{Q})$ is also an E_n -space

Exercise 9: Show that $\pi_0 L(\mathcal{E}, \mathcal{Q}) = L_0(\mathcal{E}, \mathcal{Q})$.

Deduce that $L(\mathcal{E}, \mathcal{Q})$ is a grouplike E_n -space and hence a connective spectrum

Exercise 10: Show that $|[n] \mapsto \text{Fm}(\mathcal{P}_n(\mathcal{E}, \mathcal{Q}))|$ is contractible.

Hint: use extension by 0's to define an extra degeneracy.

Definition: Let $\text{Met}(\mathcal{E}, \mathcal{Q})$ be the Poincaré subcategory of $\mathcal{P}_1(\mathcal{E}, \mathcal{Q})$ on objects of the form $\begin{array}{ccc} & W & \\ \circ \swarrow & & \searrow \\ & & X \end{array}$

concretely $\mathcal{P}^{\text{met}}(L \rightarrow X) = \text{fib}(\mathcal{Q}(X) \rightarrow \mathcal{Q}(L)) + \mathcal{D}_{\text{met}}(L \rightarrow X) = \{ \mathcal{D}(L \rightarrow X) \rightarrow \mathcal{D}_X X \}$ earlier analysis $\Rightarrow \mathcal{P}_n(\text{Met}(\mathcal{E}, \mathcal{Q})) = \{ \text{lagrangians } (L \rightarrow X, \eta) \}$ is Poincaré objects of \mathcal{E}, \mathcal{Q}

Prop: $(\mathcal{E}, \mathcal{Q}) \xrightarrow{i} \text{Met}(\mathcal{E}, \mathcal{Q}) \xrightarrow{t} (\mathcal{E}, \mathcal{Q})$ is a split Poincaré-Vershik sequence.
 $(L \rightarrow X) \mapsto X$
 $y \mapsto (y \rightarrow 0)$
 i.e. fibre + cofibre sequence in $\text{Cat}_{\infty}^{\mathcal{P}}$, s.t. t (just as functor of stable ∞ -cats) admits left + right adjoints.

proof: $\varphi^{\text{met}}(i(y)) = \varphi^{\text{met}}(y \rightarrow 0) = \text{fib}(\mathcal{Q}(0) \rightarrow \mathcal{Q}(y)) = \Omega \mathcal{Q}(y) \quad \checkmark$

$\varphi^{\text{met}}(L \rightarrow X) = \text{fib}(\mathcal{Q}(X) \rightarrow \mathcal{Q}(L)) \rightarrow \mathcal{Q}(X)$ defines the datum of form. functor $t(\mathcal{D}_{\text{met}}(L \rightarrow X)) \simeq \mathcal{D}_X(X)$

split PV proj. is implied by: t has fully faithful left adjoint l and $\varphi \simeq \varphi^{\text{met}} \circ l$
 here: $l(X) = (0 \rightarrow X) \quad \checkmark \quad \varphi^{\text{met}}(\mathcal{Q}(X)) = \text{fib}(\mathcal{Q}(X) \rightarrow \mathcal{Q}(0)) = \mathcal{Q}(X)$
 the adjoint need not be Poincaré functors. $\varphi =$ left Kan ext. from met

(Ponicki, Lurie)

(split)

Theorem: $L: \text{Cat}_\omega^P \rightarrow \mathcal{D}_{\geq 0}$ sends Poincaré-Verdier sequences to fibre sequences

Proposition: $\pi_n L(\mathcal{E}, \mathcal{Q}) \cong L_0(\mathcal{E}, \mathcal{Q}^{\text{op}})$

proof: Induction over n . $n=0$ ✓ (Exercise 9)

$$n \geq 1: \pi_n L(\text{Met}(\mathcal{E}, \mathcal{Q})) \rightarrow \pi_n(L(\mathcal{E}, \mathcal{Q})) \rightarrow \pi_{n-1}(L(\mathcal{E}, \mathcal{Q}^{\text{op}})) \rightarrow \pi_{n-1} L(\text{Met}(\mathcal{E}, \mathcal{Q}))$$

ind. Hs $L_0(\mathcal{E}, \mathcal{Q}^{\text{op}})$ on underlying categories $\text{Fun}(\mathbb{Z}_n^{\text{op}}, \text{Fun}(\Delta, \mathcal{C})) \cong \text{Fun}(\Delta, \text{Fun}(\mathbb{Z}_n^{\text{op}}, \mathcal{C}))$

suffices thus to prove $L(\text{Met}(\mathcal{E}, \mathcal{Q})) \cong *$.

$$L(\text{Met}(\mathcal{E}, \mathcal{Q})) = \left| [n] \mapsto P_n(\text{Met}(\mathcal{E}, \mathcal{Q})) \right|$$

$$\text{Observation: } P_n \text{Met}(\mathcal{E}, \mathcal{Q}) \cong \text{Met}(P_n(\mathcal{E}, \mathcal{Q}))$$

alg. Thom isomorphism

$$\text{Proposition: } P_n(\text{Met}(\mathcal{E}, \mathcal{Q})) \cong \text{Fm}(\mathcal{E}, \mathcal{Q}^{\text{op}})$$

(Ponicki)

red. 2 exercise
obj. $(\mathcal{E}, \mathcal{Q})$

morph. $F: \mathcal{E} \rightarrow \mathcal{E}^1 + n=1$
 $\eta: \mathcal{Q} \rightarrow \mathcal{Q}^{\text{op}}$
has left + right adjoints, l, r

$$\begin{array}{ccccc} x & \leftarrow & w & \rightarrow & x^1 & p_1 \\ \downarrow & & \downarrow & & \downarrow & \\ y & \leftarrow & v & \rightarrow & y^1 & p_2 \end{array}$$

reason: the forgetful functor $\text{Cat}_\omega^P \xrightarrow{U} \text{Cat}_\omega^{\text{h}}$ has left + right adjoints, l, r

$$+ r(U(\mathcal{E}, \mathcal{Q})) = \text{Met}(\mathcal{E}, \mathcal{Q}^{\text{op}})$$

$(\mathcal{Q}_n^{\text{op}})^{\text{met}}$ = fibre of pullbacks of \mathcal{Q} 's
||
 $(\mathcal{Q}^{\text{met}})_n$ = pullback of fibre of \mathcal{Q} 's.

$$\begin{aligned} \text{Hence: } P_n(\text{Met}(\mathcal{E}, \mathcal{Q})) &= \text{Map}_{\text{Cat}_\omega^P}((\mathcal{E}^{\text{op}}, \mathcal{Q}^{\text{op}}), \text{Met}(\mathcal{E}, \mathcal{Q})) = \text{Map}_{\text{Cat}_\omega^{\text{h}}}((\mathcal{E}^{\text{op}}, \mathcal{Q}^{\text{op}}), rU(\mathcal{E}, \mathcal{Q})) \\ &= \text{Map}_{\text{Cat}_\omega^{\text{h}}}(\mathcal{U}(\mathcal{E}^{\text{op}}, \mathcal{Q}^{\text{op}}), \mathcal{U}(\mathcal{E}, \mathcal{Q})) = \text{Fm}(\mathcal{E}, \mathcal{Q}^{\text{op}}) \end{aligned}$$

the equivalence is implemented by an alg. surgery as we will see tomorrow. □ Prop.

$$\text{Thus: } L(\text{Met}(\mathcal{E}, \mathcal{Q})) = \left| [n] \mapsto P_n(\text{Met}(\mathcal{E}, \mathcal{Q})) \right|$$

$$= \left| [n] \mapsto P_n(\text{Met}(P_n(\mathcal{E}, \mathcal{Q}))) \right| = \left| [n] \mapsto \text{Fm}(P_n(\mathcal{E}, \mathcal{Q}^{\text{op}})) \right| = *$$

Exercise 10

• Thus, $\pi_n L(\mathcal{E}, \mathcal{Q})$ has a uniform generators and relations description (Poincaré duality modulo null cobordant ones)

Construction: Let R be a ring, M an invertible module w/ involution over M .

$$\text{Define: } W^{\text{op}}(R; M) \rightarrow L_0(\mathcal{D}^{\text{op}}(R); \mathcal{Q}_M^{\text{op}}) \rightarrow L_0(\mathcal{D}^{\text{op}}(R); \mathcal{Q}_M^{\text{op}})$$

$$\text{by: } [P, b] \mapsto [P[0], b \in \Omega^{\text{op}} \mathcal{Q}_M^{\text{op}}(P)] \text{ using } \Omega^{\text{op}} \mathcal{Q}_M^{\text{op}}(P) = \text{Hom}_{R \otimes R}(P \otimes P, M)^{\mathbb{Z}_2}$$

$$[P, b, q] \mapsto [P[0], (b, q) \in \pi_0 \Omega^{\text{op}} \mathcal{Q}_M^{\text{op}}(P)]$$

$$\text{Likewise: } W^{\text{q}}(R; M) \rightarrow L_0(\mathcal{D}^{\text{op}}(R); \mathcal{Q}_M^{\text{q}})$$

$$\text{using } \pi_0 \Omega^{\text{q}} \mathcal{Q}_M^{\text{q}}(P) \cong \Omega^{\text{q}} \mathcal{Q}_M^{\text{q}}(P) = \text{Hom}_{R \otimes R}(P \otimes P, M)_{\mathbb{Z}_2}$$

$$\rightarrow L_0(\mathcal{D}^{\text{op}}(R); \mathcal{Q}_M^{\text{q}})$$

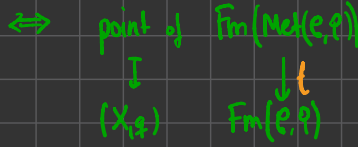
Keep in mind the question: What is the relation between $W^*(P)$ and $L_0(\mathcal{D}^p(P), \mathcal{D}_p^S)$

Algebraic Surgery Idea: simplify representatives of $L_0(\mathcal{D}^p(P))$ without changing their cobordism class.

Definition: A surgery datum on a form (X, η) consists of a pair (S, γ) where

• $S \xrightarrow{f} X$ is a map, and

• $\gamma: f^*(\eta) \sim 0$ in $\Omega^p(S)$



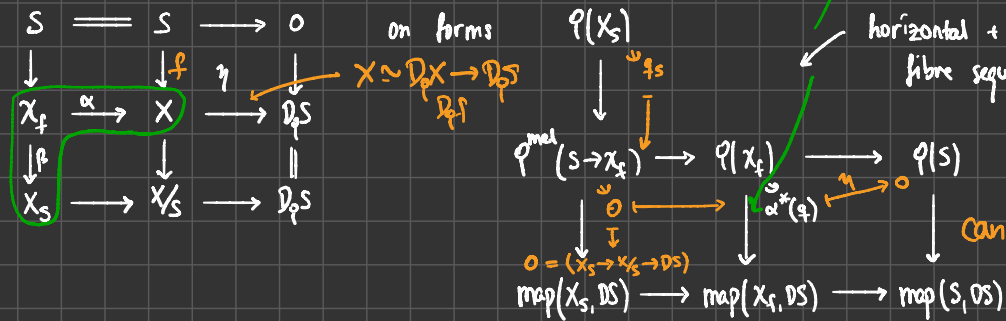
Proposition: there is a Poincaré functor $\chi: \text{Met}(e, \eta) \rightarrow \mathcal{P}_2(e, \eta)$

sending a surgery datum to the trace of its surgery

inducing an equivalence $\text{Fm}(\text{Met}(e, \eta)) \times \mathcal{P}_n(e, \eta) \rightarrow \mathcal{P}_n(\mathcal{P}_2(e, \eta))$



Sketch: the span associated to $\chi((S \xrightarrow{f} X, \eta))$ is constructed as follows:



This is the effect of the functor χ on forms. We indicate that it restricts to an alg. Thom isom.

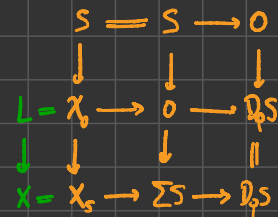
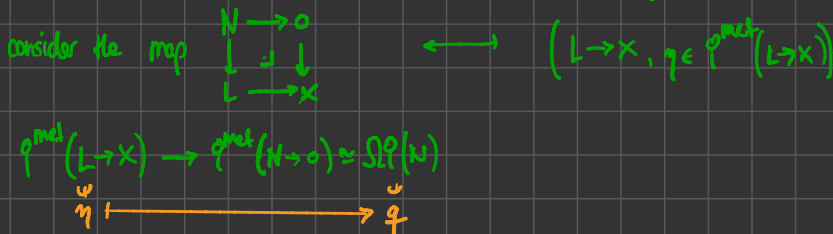
equivalence $\text{Fm}(\text{Met}(e, \eta)) \times \{0\} \simeq \mathcal{P}_n(\mathcal{P}_2(e, \eta) \times \{0\})$

$\underbrace{\hspace{10em}}_{\text{Fm}(e, \mathcal{D}_p \eta)}$

$\underbrace{\hspace{10em}}_{\mathcal{P}_n(\text{Met}(e, \eta))}$

Exercise 11: Show that the underlying object of the surgery output of $(S \rightarrow 0, \eta \in \Omega^p(S))$ is given by $\text{cofib}(S \xrightarrow{\eta} \mathcal{D}_p S)$.

$(S, \eta \in \Omega^p(S)) \longmapsto (S \rightarrow 0, \eta) \longmapsto \text{output of surgery on } 0$



Next goal: Show that the map $W^s(R; M) \rightarrow L_0(\mathcal{D}^s(R); \mathcal{P}_M^{\text{proj}})$ is an isomorphism.

in particular, need to argue that, up to cobordism, Poincaré objects for $\mathcal{P}_M^{\text{proj}}$ can be assumed to be of the form $\mathcal{P}[0]$ for $\mathcal{P} \in \text{Proj}(R)$.

Lemma: Let R, M as before. Let $X \in \mathcal{D}^s(R)$. TFAE X is m -conn. $\pi_i(X) = 0$ if $i < m$

$\pi_i(X)$
 $\mathcal{P}_i \in \text{Proj}(R)$ string in hom. degree i

1) X is m -connective and $D_n X$ is $(-n)$ -connective

2) X is repr. by a chain complex $0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_{m+1} \rightarrow P_m \rightarrow 0$

proof: step 1: X repr. by some finite chain complex C , i.e. C_i is f.g. proj. and almost all $C_i = 0$
 2) \Rightarrow 1)

step 2: since X is m -conn. \Rightarrow achieve $C_i = 0$ for $i < m$

step 3: DX repr. by applying $\text{Hom}_R(-, M)$ to the representing chain complex C

step 4: since DX is $(-n)$ connective \Rightarrow achieve $\text{Hom}_R(C, M)_i = 0$ for $i < -n$

step 5: $X \simeq D^2 X$ is now of the claimed shape. □

eg. $\mathcal{P}_M^{\text{proj}}, \mathcal{P}_M^{\text{proj}}, \mathcal{P}_Z^{\text{proj}}$

Definition: R, M as always. \mathcal{P} M -compatible + assume $\mathcal{P}(P)$ is an EM spectrum in degree 0 for any

$P \in \text{Proj}(R)$. Let $W(\text{Proj}(R), \mathcal{P}) = \left(\begin{array}{l} \text{isom. classes of unim.} \\ \mathcal{P}\text{-forms on f.g. proj.} \end{array} \right) \oplus \left(\text{metabolic forms} \right)$ sometimes called "strictly" metabolic

Proposition: Under the above assumptions, the canonical map

$W(\text{Proj}(R), \mathcal{P}_0) \rightarrow L_0(\mathcal{D}^s(R), \mathcal{P})$ is an isomorphism.

proof: surjectivity: Let (X, \mathcal{q}) be a Poincaré object for \mathcal{P} .

if X is connective $\Rightarrow X \simeq^{\mathcal{q}} DX$ is also connective.

hence $X \simeq \mathcal{P}[0]$ by previous lemma. \checkmark

in general, X is $(-k)$ connective for some $k \geq 0$ (or $X=0$) assume $-k = \min \{ \pi_i(X) \neq 0 \} < 0$

By the lemma $\Rightarrow \pi_k(X) \leftarrow \mathcal{P}$ f.g. proj. $\rightsquigarrow \mathcal{P}[-k] \xrightarrow{f} X$ in $\mathcal{D}^s(R)$

now: $\mathcal{P}[\mathcal{P}[-k]]$ is k -connective $\Rightarrow \pi_0 \mathcal{P}[\mathcal{P}[-k]] = 0$.

$\Rightarrow f^*(\mathcal{q}) \sim 0$ in $\Omega^{\mathcal{P}} \mathcal{P}[\mathcal{P}[-k]]$ so any nullhtpy η makes $(\mathcal{P}[-k] \xrightarrow{f} X, \mathcal{q})$ a surgery datum.

Exercise 12: the surgery output (X', \mathcal{q}') satisfies that X' is $(-k+1)$ -connective.

Injectivity: Assume given $P \in \text{Proj}(R)$ and $q \in \mathcal{Q}(R)$, and assume that its image in $L_0(\mathcal{D}^p(R), \mathcal{Q})$ vanishes, i.e. $\exists L \in \mathcal{D}^p(R)$, a map $f: L \rightarrow P[0]$ and a null-homotopy $g: f^*q \sim 0$ such that $L \rightarrow P[0] \xrightarrow{g} \mathcal{D}^p[0] \rightarrow \mathcal{Q}$ is a fibre sequence.

abstractly, $(L \rightarrow P, g) \in \text{Pn}(\text{Met}(\mathcal{D}^p(R), \mathcal{Q}))$ being $(P, q) \in \text{Pn}(\mathcal{D}^p(R), \mathcal{Q})$

Claim: Using surgeries, we can make L (-1) -connective.

Exercise 12: then L is represent. by a complex $0 \rightarrow Q \rightarrow N \rightarrow 0$ degree $(N) = -1$.

proof: assume $-k = \min\{\pi_i(L) \neq 0\}$. Pick a surj. $Q[-k] \rightarrow \pi_{-k}(L)$ and

the ass. map $Q[-k] \rightarrow L$. The diagram $Q[-k] \rightarrow 0$ commutes as
 $\downarrow \quad \downarrow$
 $L \rightarrow P$
 $\varphi^{\text{met}}(Q[-k] \rightarrow 0) = Q^{\varphi^{\text{met}}}(Q[-k])$ is $(k-1)$ -connective. $\text{map}_R(Q[-k], P) = \text{Ext}_R^k(Q, P) = 0$.

Exercise 13: surgery output ass. to the map $\begin{pmatrix} Q[-k] \\ \downarrow \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} L \\ \downarrow \\ P \end{pmatrix}$ in $\text{Met}(\mathcal{D}^p(R))$ is of the form $L' \rightarrow P$ with L' $(-k+1)$ -connective

Inductively $\leadsto L$ can be assumed (-1) -connective. Writing L as $\begin{pmatrix} 0 \rightarrow Q \rightarrow N \rightarrow 0 \end{pmatrix}$ Ex. 12
 We find a map $N[-1] \xrightarrow{(*)} L$ whose cofibre is $Q[0]$.

Consider the map $\begin{matrix} N[-1] & \longrightarrow & L \\ \parallel & & \downarrow \\ N[-1] & \longrightarrow & P \end{matrix}$ in $\text{Met}(\mathcal{D}^p(R))$. $\varphi^{\text{met}}(N[-1] \rightarrow L) = 0$

\leadsto can do surgery. Output is a Lagrangian of the form $Q[0] \rightarrow P \oplus (N \oplus DN)$

where the form on $P \oplus (N \oplus DN)$ is (q, q') with q' metabolic ($DN \rightarrow N \oplus DN$ is a Lagrangian for q')

\Rightarrow in $W(\text{Proj}(R); \mathcal{Q})$ have: $[P, q] = [P, q] + [N \oplus DN, q'] = [P \oplus (N \oplus DN), q \oplus q'] = 0$ since $Q[0]$ is a Lagrangian here.

Corollary: $L_0(\mathcal{D}^p(R), \mathcal{Q}^{\text{ss}}) \cong W^s(R; M)$ and $L_0(\mathcal{D}^p(R), \mathcal{Q}^{\text{sa}}) \cong W^q(R; M)$

Definition: Let $\varphi: \mathcal{D}^p(R)^{\text{op}} \rightarrow \mathcal{S}_p$ be an M -compatible Poincaré structure, associated to

a map $N \xrightarrow{\alpha} M^{\text{ts}_2}$ of R -modules. We say that φ is

1) m -quadratic if N is m -connective

$\pi_i(N) = 0$ for $i < m$

2) r -symmetric if the fibre of α is $(-r)$ -truncated.

$\pi_i(\text{fibre}) = 0$ for $i > -r$

Exercise 44: 1) φ is m -quadratic $\Leftrightarrow L_{\varphi}(X)$ is $(m+k)$ -connective whenever DX is k -connective

2) φ is r -symmetric $\Leftrightarrow \text{fib}(\varphi(X) \rightarrow \varphi^S(X))$ is $(-r-k)$ -truncated whenever X is k -connective.

Next goal: make qualitative statements about the maps

$$L(\mathcal{D}^p(R), \varphi_M^q) \longrightarrow L(\mathcal{D}^p(R), \varphi) \longrightarrow L(\mathcal{D}^p(R), \varphi_M^S)$$

for an M -compatible Poincaré structure φ

Recall: we calculated $L_0(\mathcal{D}^p(R), \varphi)$
in terms of de Rham forms
if $\varphi(p) = \text{EM-spectrum in degree } 0$

Thm: Let φ be an m -quadr. M -comp. Poincaré structure on $\mathcal{D}^p(R)$. Then

$$\pi_n L(\mathcal{D}^p(R), \varphi_M^q) \longrightarrow \pi_n L(\mathcal{D}^p(R), \varphi) \quad \text{is } \begin{cases} \text{an isom. for } n \leq 2m-3 \\ \text{surjective for } n \leq 2m-2 \end{cases}$$