Exercise 14: 1) $ϙ$ is m-quadratic $\Leftrightarrow$ $L_ϙ(X)$ is $(m+k)$-connective whenever DX is k-connective

2) $ϙ$ is r-symmetric $\Leftrightarrow$ $\mathrm{fib}(ϙ(X) \to ϙ^s(X))$ is $(-r-k)$-truncated whenever X is k-connective.

Next goal: make qualitative statements about the maps

$$L(\mathcal{D}^p(R), ϙ^q_M) \longrightarrow L(\mathcal{D}^p(R), ϙ) \longrightarrow L(\mathcal{D}^p(R), ϙ^s_M)$$

for an M-compatible Poincaré structure $ϙ$

Recall: we calculated $L_0(\mathcal{D}^p(R), ϙ)$ in terms of classical forms if $ϙ(P)$ = EM-spectrum in degree 0

Thm: Let $ϙ$ be an m-quadr. M-comp. Poincaré structure on $\mathcal{D}^p(R)$. Then

$$\pi_n L(\mathcal{D}^p(R), ϙ^q_M) \xrightarrow{\circledast} \pi_n L(\mathcal{D}^p(R), ϙ) \quad \text{is} \begin{cases} \text{an isom. for } n \le 2m-3 \\ \text{surjective for } n \le 2m-2 \end{cases}$$

proof Idea: Step 1: Show that elements of both sides are represented by (X, q) sth.

X is $(-k)$-connective if $n = 2k, 2k-1$ (assuming $n \le 2m-2$) $\Rightarrow$ $k \le m-1$ ie $m > k$

↳ $X = P[-k]$ if $n = 2k$ and X repr. by a complex conc. in degrees $[-k, -k+1]$ if $n = 2k-1$.

proof: let $l < -k$. have a fibre sequence

$$(\dagger) \quad \underbrace{\Omega^n \underbrace{\mathrm{map}_{R\otimes R}(P[l]\otimes P[l], M)}_{\ge -2l}{}^{hC_2}}_{\ge -2l-n \;>\; 2k-n \;\ge 0} \longrightarrow \underbrace{\Omega^n ϙ(P[l])}_{\text{is connected.}} \longrightarrow \Omega^n L_ϙ(P[l]) \simeq \underbrace{\Omega^n \mathrm{map}_R(P[l], N)}_{\substack{\ge m-l-n \\ > k+k-n \,=\, 2k-n \,\ge 0}}$$

can perform surgery on $P[l] \to X$, similarly as before we inductively see Step 1.

Step 2: For $(-k)$-conn. Poincaré objects X for $ϙ^q_M$, the map $\Omega^n ϙ^q_M(X) \to \Omega^n {}^{q}ϙ(X)$ is π_0 surjective.

proof: case $n = 2k$: $X = P[-k]$. fibre sequence $(\dagger)$ $\Rightarrow$ suffices to see that $\Omega^n \mathrm{map}_R(P[-k], N)$ is connected. It is $(m+k-n)$-conn. + $(m+k-n) \overset{m>k}{>} 2k-n \ge 0$ ✓

similarly for $n = 2k-1$ $\Rightarrow$ surjectivity of $\circledast$

Step 3: similar argument for Lagrangians wrt $ϙ^q_M$ and $ϙ$ in a $(-k)$-connective Poincaré object for $\Omega^n ϙ^q_M$. $\Rightarrow$ injectivity of $\circledast$.

∎

Corollary: $L_n(\mathcal{D}^p(R), \varphi^q_M) \longrightarrow L_n(\mathcal{D}^p(R), \varphi^{gq})$ is an isom for $n \le 1$, surj. for $n=2$

$L_n(\mathcal{D}^p(R), \varphi^q_M) \longrightarrow L_n(\mathcal{D}^p(R), \varphi^{gs})$ is an isom. for $n \le -3$, surj. for $n=-2$

In fact the functor $\Omega: \mathcal{D}^p(R) \to \mathcal{D}^p(R)$ extends to an equivalence of Poincaré ∞-categories

$$(\mathcal{D}^p(R), \varphi^{\ge m}_M) \longrightarrow (\mathcal{D}^p(R), \Omega^2 \varphi^{\ge (m+1)}_{-M})$$

Exercise: prove this and deduce that $L(\mathcal{D}^p(R), \varphi^q_M)$ and $L(\mathcal{D}^p(R), \varphi^s_M)$ are 4-periodic

$\Rightarrow L_{-2}(\mathcal{D}^p(R), \varphi^{gs}_M) = L_0(\mathcal{D}^p(R), \varphi^{gev}_{-M}) \cong W^{-ev}(R;M)$

Next goal: What can one say about $L_n(\mathcal{D}^p(R), \varphi^{gs}_M)$ for $n > 0$?

It will be convenient to give a technical definition of L-groups with connectivity estimates on objects + Lagrangians.

Exercise: $L_n^{a,b}$ can equiv. be defined by saying: X is $\lceil \frac{-n-a}{2} \rceil$-connect. + Lagr. $L \to X$ satisfy L is $\lceil \frac{-n-1-b}{2} \rceil$-conn. & $\mathrm{fib}(L \to X)$ $\lfloor \frac{-n-1-b}{2} \rfloor$-conn.

Definition: R, M as always. $n, a, b \in \mathbb{Z}$, $a, b \ge -1$, $b \ge a-1$, $n+a \equiv 0\ (2)$, φ M-comp. Poincaré str.

Let $L_n^{a,b}(\mathcal{D}^p(R), \varphi) = \Big\{$ equiv. classes of Poincaré objects (X,q) s.th. X can be repr. by a complex of "length a" concentr. necessarily in degrees $[\frac{-n-a}{2}, \frac{-n+a}{2}]$ $\Big\}$ / those that admit a Lagrangian L concentrated in degrees $[\lceil \frac{-n-1-b}{2} \rceil, \lceil \frac{-n-1+b}{2} \rceil]$

Step 1+3 of the last theorem can be phrased that for suitable a,b,n,m, the map $L_n^{a,b}(\mathcal{D}^p(R), \varphi) \longrightarrow L_n(\mathcal{D}^p(R), \varphi)$ is an isom / surjective.

and Step 2) that $L_n^{a,b}(\mathcal{D}^p(R), \varphi^q_M) \to L_n^{a,b}(\mathcal{D}^p(R), \varphi)$ is an isom / surj.

"short symmetric L-theory of Ranicki"

Thm: $L_n(\mathcal{D}^p(R), \varphi^{gs}_M) \cong L_n^{short}(R;M) := L_n^{n,n+1}(\mathcal{D}^p(R), \varphi^s_M)$ for $n \ge 0$. $\Rightarrow$ repr. by chain compl. in the interval $[-n, 0]$

proof: 1) elements of $L_n(\mathcal{D}^p(R), \varphi^{gs}_M)$ can be represented by $(-n)$-conn. complexes

2) on $(-n)$-conn. complexes, $\pi_0(\Omega^n \varphi^{gs}_M(X)) \xrightarrow{\cong} \pi_0(\Omega^n \varphi^s_M(X))$ is an isom.

3) if a $(-n)$-conn. Poincaré obj. admits a Lagrangian, it admits one which is $(-n-1)$-connective concentrated in $[-n-1, 0]$

1)+3) follow from surgery, using that φ^{gs}_M is 0-quadratic.

2) follows from the observation that φ^{gs}_M is 1-symmetric.

⊛ every module has a proj. resolution of length $\leq d$.

Thm: let R be a Noetherian ring of finite global dim d, M as always. Let φ be a M-comp.

and r-symmetric Poincaré str. Then the map

$$L_n(\mathcal{D}^p(R),\varphi) \longrightarrow L_n(\mathcal{D}^p(R),\varphi^s_M) \text{ is } \begin{cases} \text{an isom for } & n \geq d-2r+3 \\ \text{injective for } & n \geq d-2r+2 \end{cases}$$

To prove this, it is worth to set the stage using t-structures.

$\mathcal{C} = \mathcal{D}^p(R)$ R Noeth. of fin. global dim.
t-structure is the restriction of the usual Postnikov t-str. on $\mathcal{D}(R)$.

Suppose $\mathcal{C}$ is a small $\mathcal{A}$. ∞-category equipped w/ a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$

given a duality equiv. $D: \mathcal{C}^{op} \xrightarrow{\simeq} \mathcal{C}$ we consider $\varphi^s_D : \mathcal{C}^{op} \to Sp$

$$X \mapsto \mathrm{map}_{\mathcal{C}}(X, DX)^{hC_2}$$

and say that a Poincaré structure φ on $\mathcal{C}$ is

- D-comp. if the duality ass. to φ is $D: \mathcal{C}^{op} \xrightarrow{\simeq} \mathcal{C}$
- r-sym. if $\forall X \in \mathcal{C}_{\geq 0}$ the fibre of the map $\varphi(X) \to \varphi^s_D(X)$ is $(-r)$-truncated

Exercise: Show that we have not redefined the notion of an r-sym. Poincaré structure.

Using Exercise, we can define $L_n^{a,b}(\mathcal{C},\varphi)$ by interpreting "X is k-conn." as "$X \in \mathcal{C}_{\geq k}$". ($\mathcal{C}_{\geq k} = \Sigma^k \mathcal{C}_{\geq 0}$)

We will now formulate a general theorem, but prove only a version of ⊛ (the general case is just more book-keeping)

Thm let $\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, D: \mathcal{C}^{op} \xrightarrow{\simeq} \mathcal{C}$, and φ^s_D as above. Let φ be an r-symmetric and D-compatible Poincaré str.

Assume that $D(\mathcal{C}_{\leq 0}) \subseteq \mathcal{C}_{\geq -d}$ for some $d \geq 0$. Let $n \in \mathbb{Z}$, $a \geq d-1$, $b \geq d$ ~~$b \geq d$~~ be integers s.th.

- $n+a \equiv 0 \ (2)$ and
- $a \geq -n+2d-2r$

Then $L_n^{a,b}(\mathcal{C},\varphi) \to L_n(\mathcal{C},\varphi)$ is an isomorphism.

Exercise: prove Thm ⊛ from this result by considering the diagram

$$\begin{tikzcd} L_n^{a,b}(\mathcal{D}^p(R),\varphi) \arrow[r, "\text{Thm}"] \arrow[d, "\varphi \text{ r-sym.}"'] & L_n(\mathcal{D}^p(R),\varphi) \arrow[d] \\ L_n^{a,b}(\mathcal{D}^p(R),\varphi^s_M) \arrow[r, "\text{Thm}"] & L_n(\mathcal{D}^p(R),\varphi^s) \end{tikzcd}$$

proof of the simple version: R Noeth. of dim d. Then the map

$$L_n^{n,n+1}(\mathcal{D}^p(R),\varphi^s_M) \longrightarrow L_n(\mathcal{D}^p(R),\varphi^s_M) \text{ is an isomorphism for } n \geq d-1$$

Ranicki: $n \geq 2(d-1)$

Thm before IIS

$L_n(\mathcal{D}^p(R),\varphi^{gs}_M) \; \cong$

surjectivity: let $[X,q] \in L_n(\mathcal{D}^p(R), \varphi^s_M)$. Recall that $\mathcal{D}^p(R)$ has a t-structure $(\mathcal{D}^p(R)_{\geq 0}, \mathcal{D}^p(R)_{\leq 0})$

and that $D_\varphi(\mathcal{D}^p(R)_{\leq 0}) \subseteq \mathcal{D}^p(R)_{\geq -d}$. Let $\tau_{\leq k} : \mathcal{D}^p(R) \to \mathcal{D}^p(R)_{\leq k}$ be the can. truncation functors.

We wish to perform surgeries on (X,q) to make X $(-n)$-connective.

Consider the map $X \to \tau_{\leq -n-1} X$ and apply $\Omega^n D_\varphi$ to obtain a map

$$W := \Omega^n D_\varphi \tau_{\leq -n-1} X \longrightarrow \Omega^n D_\varphi X \underset{\underset{q}{\smile}}{\overset{q^\#}{\simeq}} X$$

Exercise: $W \in \mathcal{C}_{\geq -d+1}$

Exercise: if one can perform surgery, then the surgery trace is can. equiv. to $\tau_{\geq -n}(X)$ and hence the surgery output is $(-n)$-connective.

Find that $\Omega^n \varphi^s_D(W)$ is (-1)-truncated $\Rightarrow$ can perform surgery $\Rightarrow$ surjectivity ✓

Injectivity is similar, performing surgery on a Lagrangian: Assume (X,q) is in the kernel of the map in question, i.e. X is $(-n)$-connective, and $\exists$ Lagrangian $(L \to X, \eta)$.

Let $N = \mathrm{fib}(L \to X)$ so that $L \simeq \Omega^{n+1} D_\varphi N$. Let

$$N' := \Omega^{n+1} D_\varphi \tau_{\leq -n-2} L \longrightarrow \Omega^{n+1} D_\varphi L \simeq N$$

$$\begin{tikzcd} N' \arrow[r] \arrow[d] \arrow[dr, phantom, "\simeq"] & L \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd}$$

view as map + see that $\Omega^\infty \varphi^{\mathrm{nmet}}(N' \to 0)$ is (-1)-truncated ∎

Corollary: Suppose R is Noetherian of global dimension 0 (i.e. R is semi simple). Then

1) $L_{2k-1}(\mathcal{D}^p(R), \varphi^{\geq m}_M) = 0$ for $k > m-1$

2) $L_{2k}(\mathcal{D}^p(R), \varphi^{gs}) \cong W_0(\mathrm{Proj}(R), \varphi^{gs})$ $k \geq -2$

recall that $\varphi^{\geq 0}_M = \varphi^{gs}$, $\varphi^{\geq 1}_M = \varphi^{gev}$, $\varphi^{\geq 2}_M = \varphi^{gq}$

Exercise: Show that $L_{2k-1}(\mathcal{D}^p(R), \varphi^q_M) = 0$, $L_{2k-1}(\mathcal{D}^p(R), \varphi^s_M) = 0$

3) Let $(P[0], q)$ be an M-valued sym. form (or quadratic, even) s.th. its image in $W_0(\mathrm{Proj}(R), \varphi^{gs}_M)$ (or gq, gev) vanishes

then $(P[0], q)$ is (strictly) metabolic.

proof: Thm says that
- $L^{0,0}_{2n}(\mathcal{D}^p(R), \varphi) \to L_{2n}(\mathcal{D}^p(R), \varphi)$ is an isom. provided $0 \geq -2n + \underset{=0}{2d} - 2r$ ✓
- $L^{-1,0}_{2n-1}(\mathcal{D}^p(R), \varphi) \to L_{2n-1}(\mathcal{D}^p(R), \varphi)$ is an isom. provided $-1 \geq -2n+1+2d-2r$ $\Leftrightarrow -2 \geq -2n-2r$

recall $\varphi^{\geq m}_M$ is $2-m$ sym. ∎

Corollary: R Noeth. of global dimension d. Then the map $L_n(\mathcal{D}^p(R), \varphi^{gq}_M) \longrightarrow L_n(\mathcal{D}^p(R), \varphi^{gs}_M)$ is an isomorphism for $n \geq d+3$

$$\begin{tikzcd} L_n(\mathcal{D}^p(R), \varphi^{gq}_M) \arrow[r] \arrow[dr, "\mathrm{Thm}"'] & L_n(\mathcal{D}^p(R), \varphi^{gs}_M) \arrow[d, "\mathrm{Thm}"] \\ & L_n(\mathcal{D}^p(R), \varphi^{s}_M) \end{tikzcd}$$

Assume R is 2-torsion free.

Exercise: let R be comm. Noeth. global dim d, $M=R$, $\sigma = \mathrm{id}$. Then the map $L_n(\mathcal{D}^p(R), \varphi^{gq}_R) \to L_n(\mathcal{D}^p(R), \varphi^{gs}_R)$ is an isomorphism for $n \geq d+1$. Hint: relate $L(\mathcal{D}^p(R), \varphi^{gq}_R) + L(\mathcal{D}^p(R), \varphi^{gq}_{-R})$ + likewise for gs.

Goal: Compute $L(\mathcal{D}^p(R), \varphi^{gs}_R)$ for R Dedekind ring with fraction field a number field

(i.e. rings of integers like $R = \mathbb{Z}$). So far, we know

true for R Dedekind, $\mathrm{char}(K) \neq 2$ ← $\mathrm{Frac}(R)$

$$L_n(\mathcal{D}^p(R), \varphi^{gs}_R) \cong \begin{cases} L_n(\mathcal{D}^p(R), \varphi^{q}_R) & \text{for } n \leq -3 \\ L_n(\mathcal{D}^p(R), \varphi^{s}_R) & \text{for } n \geq -2 \end{cases}$$

Exercise: Show this for $n = -2, -1$.