Equivalence of Liouville quantum gravity and the Brownian map

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joint with

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Overview

How does one make sense of the uniform measure on surfaces homeomorphic to the sphere?

- **Approach 1:** Random planar maps
  - Rooted in the combinatorics literature from the 1960s
- **Approach 2:** Liouville quantum gravity (LQG)
  - Rooted in the physics literature from the 1980s
- Relationship

Schramm-Loewner evolution, percolation, Eden growth model, diffusion limited aggregation

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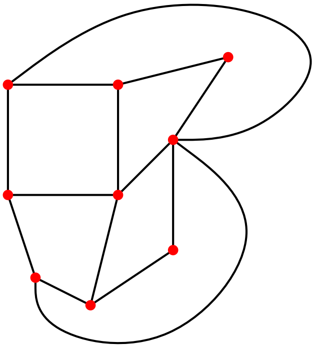
- **Donsker's invariance principle:** $S_{\lfloor tn \rfloor}/\sqrt{n}$ converges to planar Brownian motion as $n \to \infty$

Random planar maps



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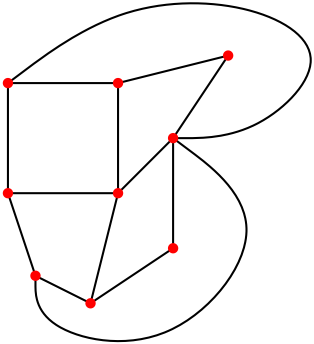

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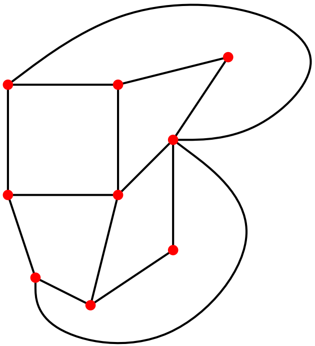

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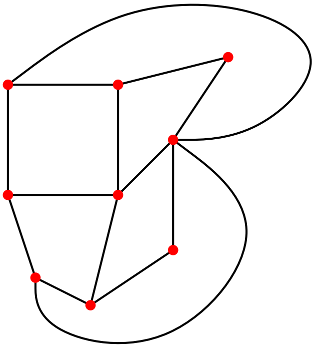

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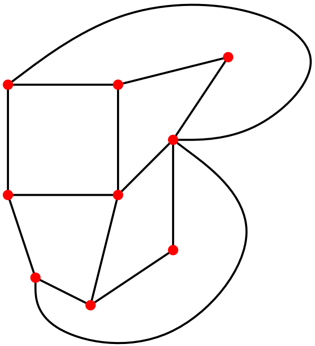

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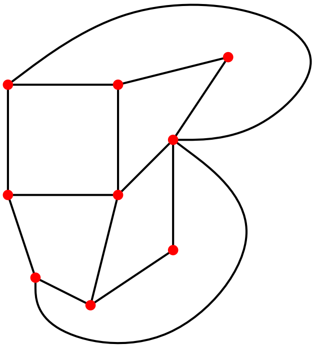

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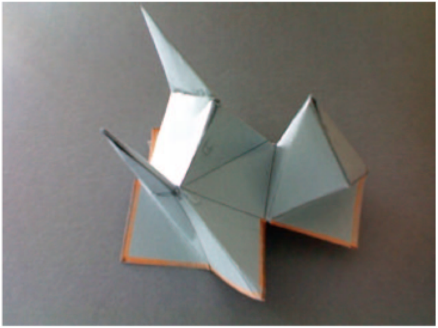
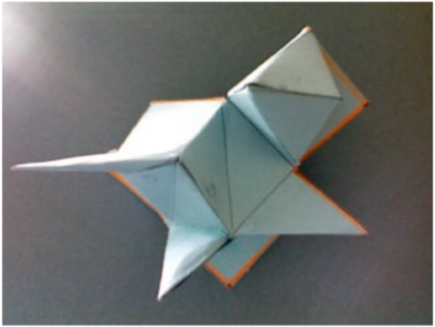
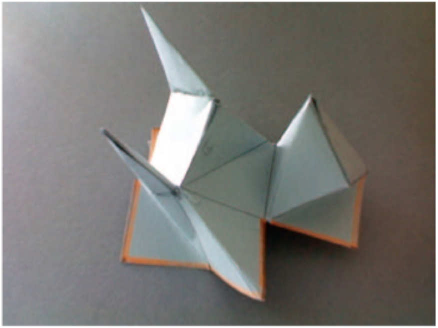
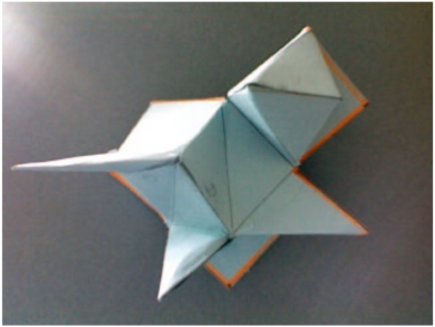
- First studied by Tutte in 1960s while working on the four color theorem
  - **Combinatorics**: enumeration formulas
  - **Physics**: statistical physics models: percolation, Ising, UST ...
  - **Probability**: "uniformly random surface," Brownian surface

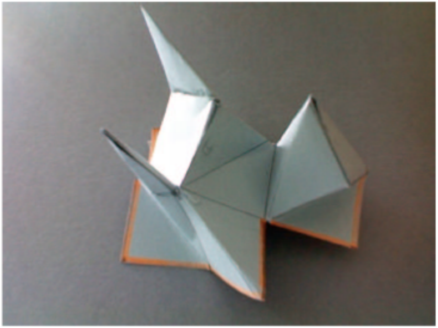
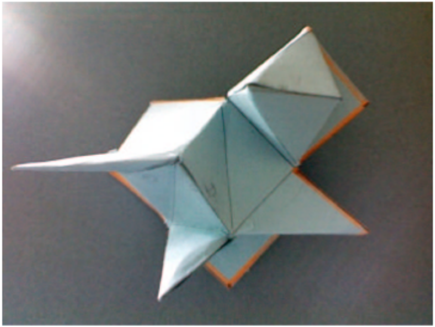








What is the structure of a typical quadrangulation when the number of faces is large?

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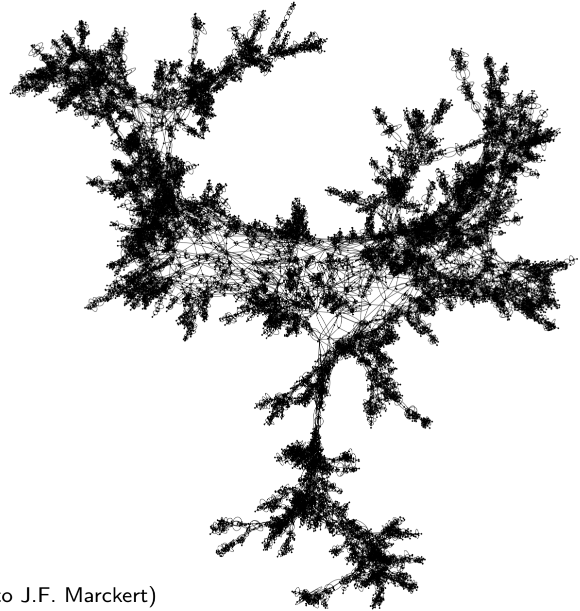
How many are there? **Tutte:**

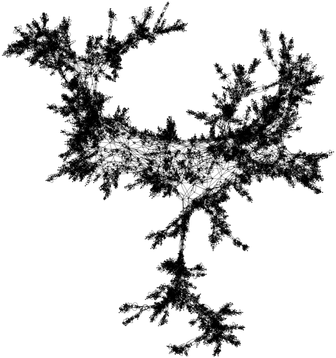
$$\frac{2 \times 3^n}{(n+1)(n+2)} \binom{2n}{n}.$$

Random quadrangulation with 25,000 faces



(Simulation due to J.F. Marckert)

Structure of large random planar maps



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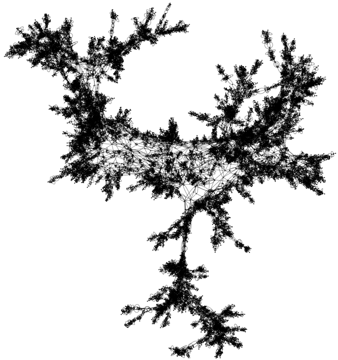

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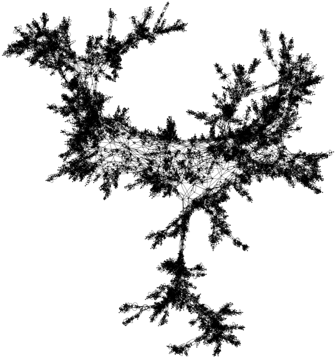

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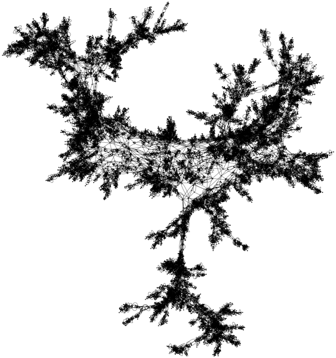

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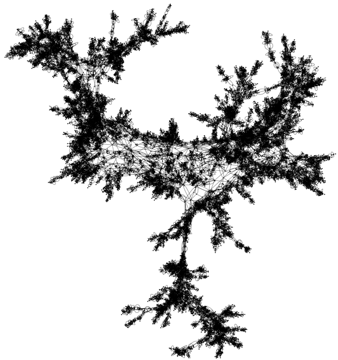

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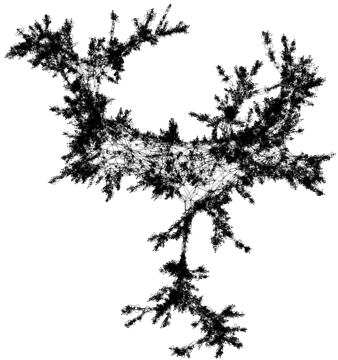

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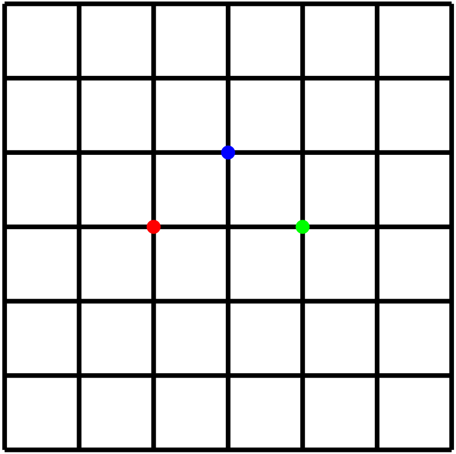
- Abstract metric measure spaces (X, d, μ)

Brownian intersection exponents



Three "random walks" on the planar grid $\mathbf{Z}^2$. Each one moves independently in each direction with equal probability.

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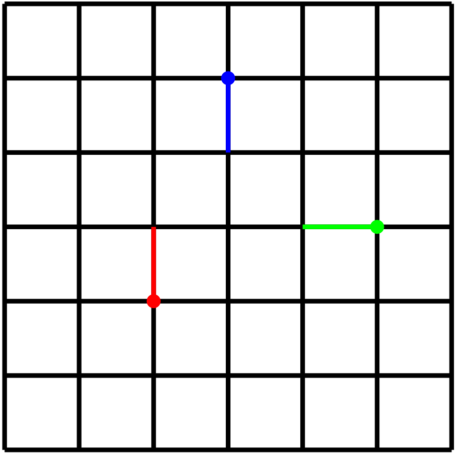

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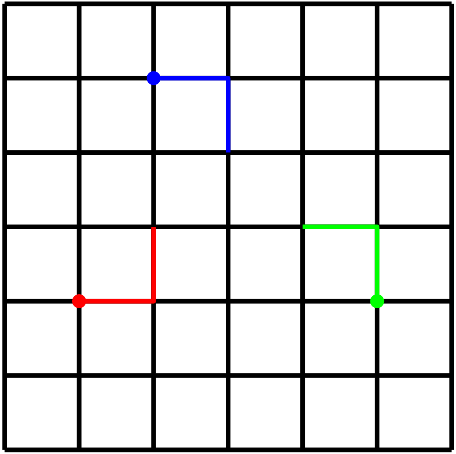

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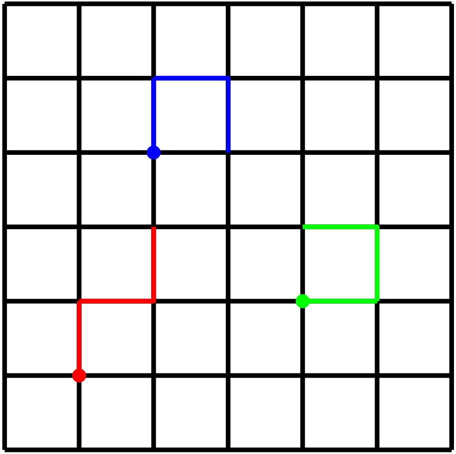

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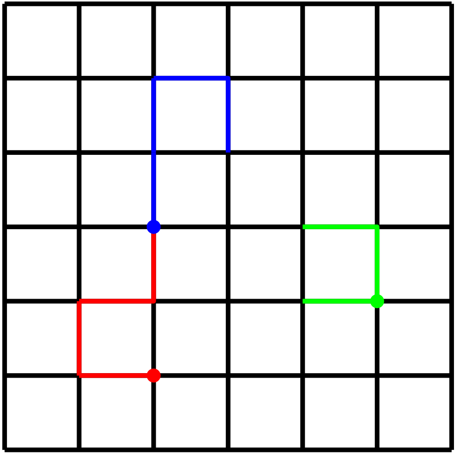

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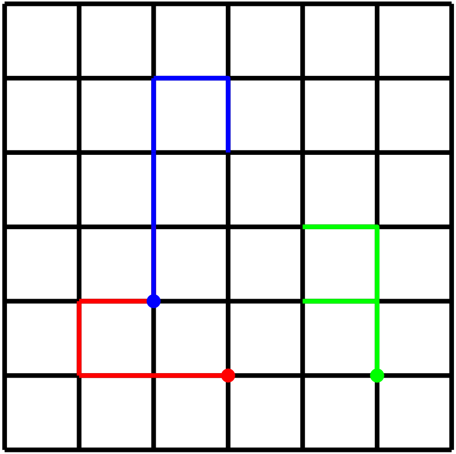

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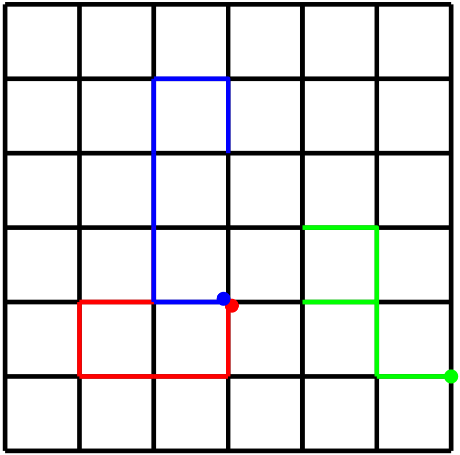

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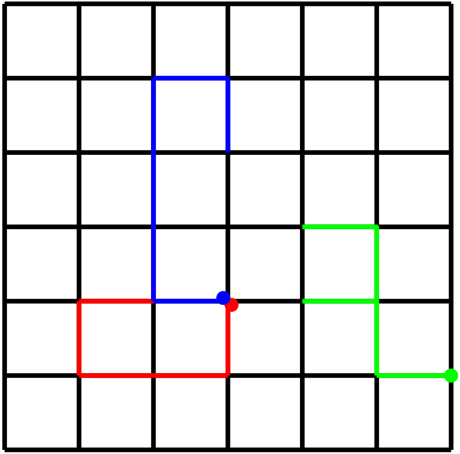

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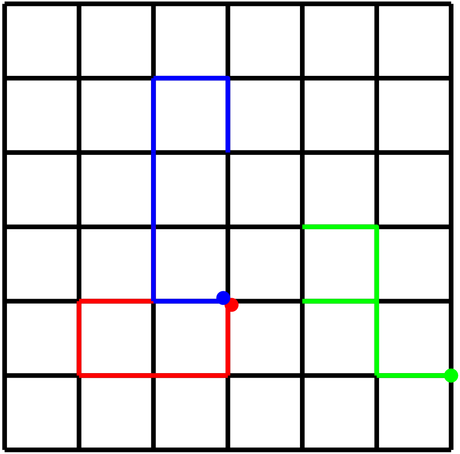

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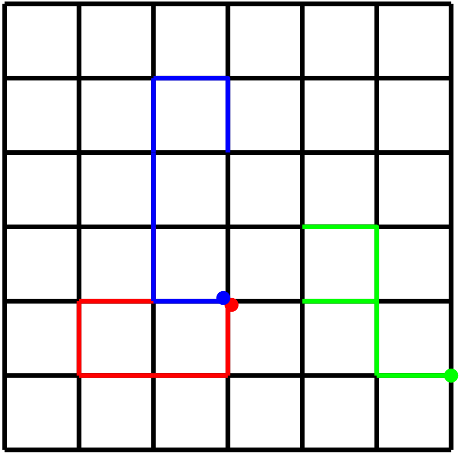

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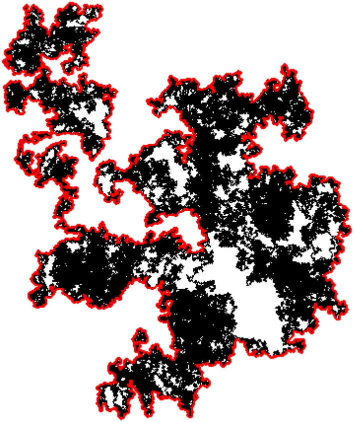

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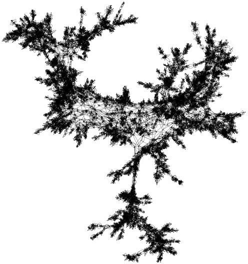

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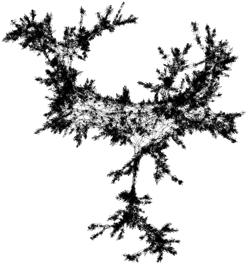

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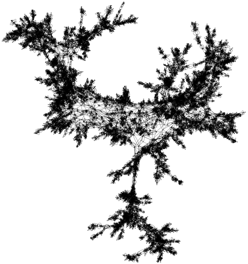

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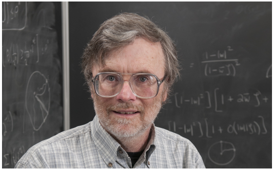

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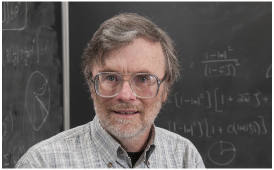
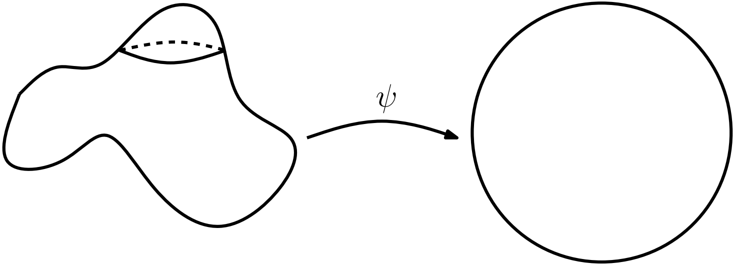
Many other examples just like this.



Picking a surface at random in the continuum

Uniformization theorem: every two-dimensional Riemannian manifold homeomorphic to the unit disk $\mathbf{D}$ can be conformally mapped to the disk.



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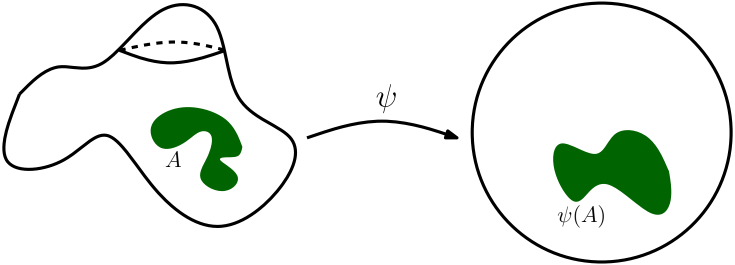

Isothermal coordinates: Metric for the surface takes the form $e^{\rho(z)}(dx^2 + dy^2)$ for some smooth function ρ where $dx^2 + dy^2$ is the Euclidean metric.

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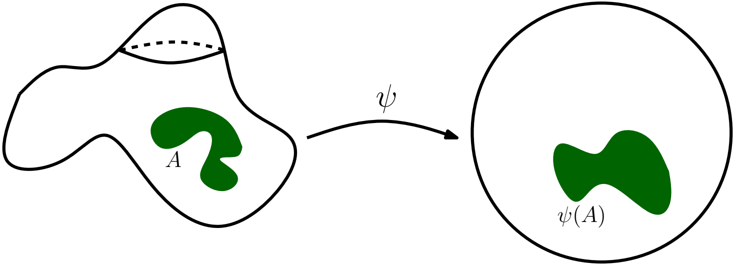

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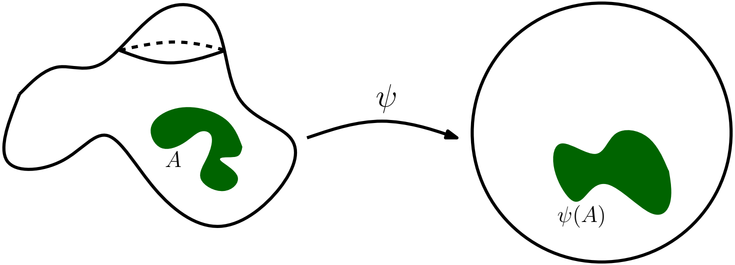

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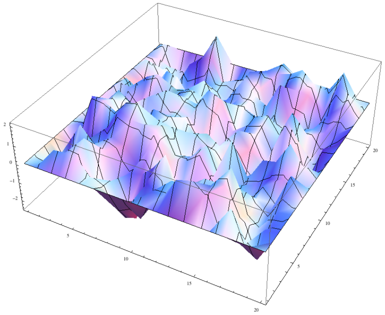
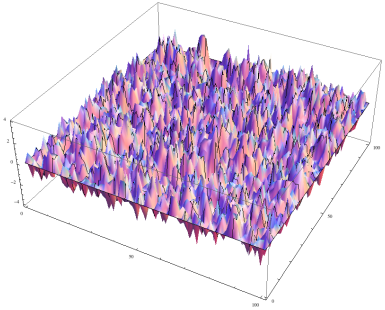
Question: Which measure on ρ? If we want our surface to be a perturbation of a flat metric, natural to choose ρ as the canonical perturbation of a harmonic function.

The Gaussian free field

- The **discrete Gaussian free field** (DGFF) is the measure on functions $h\colon D \to \mathbf{R}$ for $D \subseteq \mathbf{Z}^2$ and $h|_{\partial D} = \psi$ with density with respect to Lebesgue measure on $\mathbf{R}^{|D|}$:

$$\frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \sum_{x \sim y} (h(x) - h(y))^2 \right)$$





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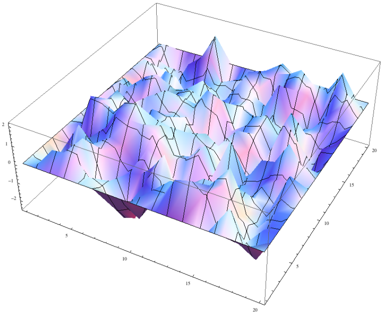

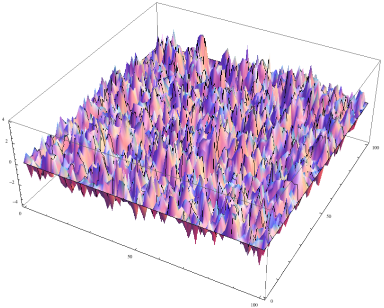

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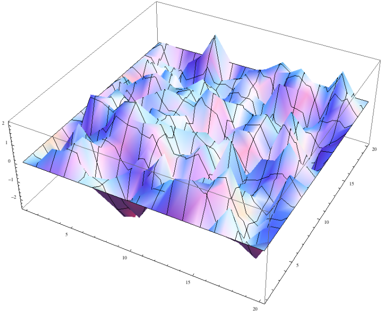
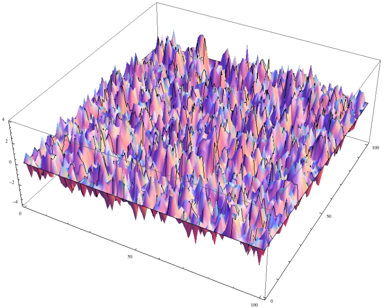
- Natural perturbation of a harmonic function
- Fine mesh limit: converges to the continuum GFF, the Gaussian field h with

$$\mathrm{cov}(h(x), h(y)) = G(x, y)$$

where G is the Green's function for Δ





The Gaussian free field

- The **discrete Gaussian free field** (DGFF) is the measure on functions $h\colon D \to \mathbf{R}$ for $D \subseteq \mathbf{Z}^2$ and $h|_{\partial D} = \psi$ with density with respect to Lebesgue measure on $\mathbf{R}^{|D|}$:

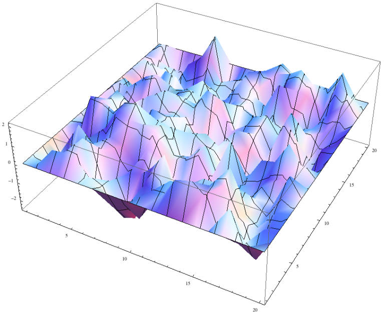
$$\frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \sum_{x \sim y} (h(x) - h(y))^2 \right)$$

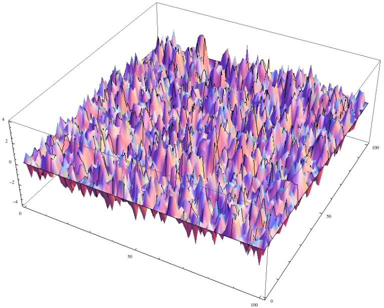
- Natural perturbation of a harmonic function
- Fine mesh limit: converges to the continuum GFF, the Gaussian field h with

$$\mathrm{cov}(h(x), h(y)) = G(x, y)$$

where G is the Green's function for Δ

- Conformally invariant and Markovian





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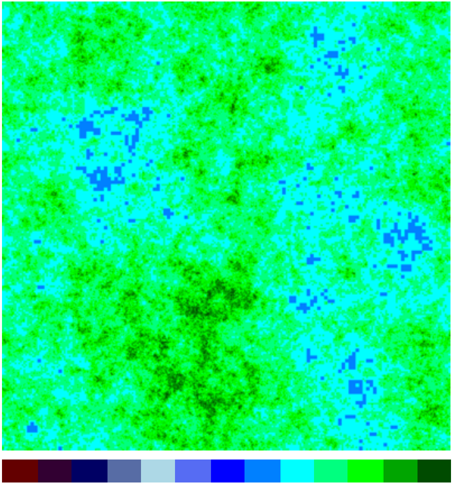

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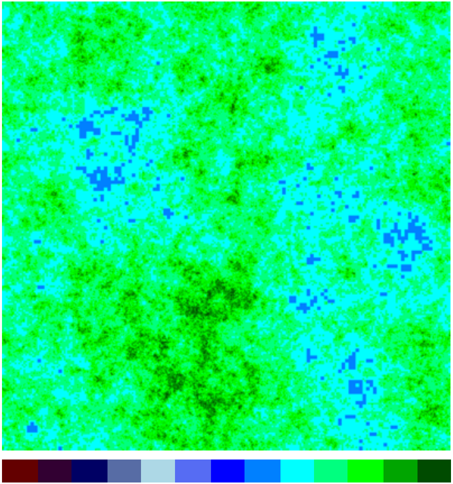

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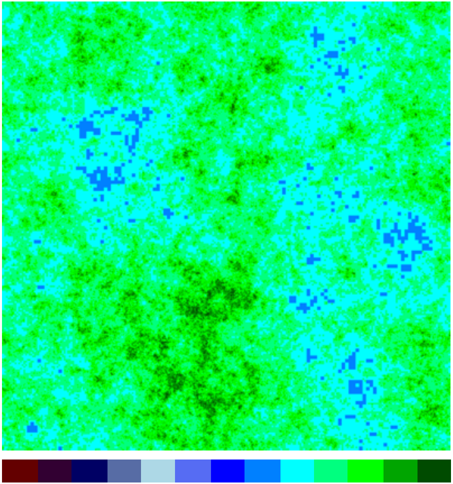

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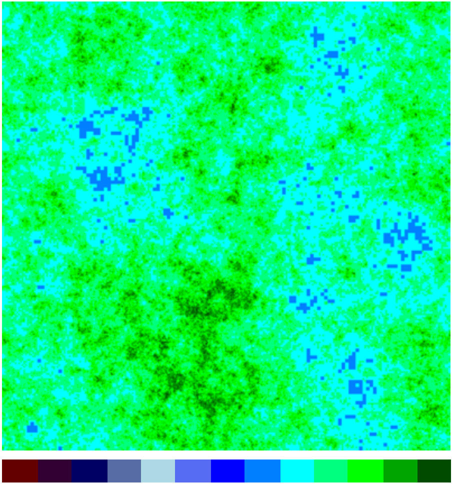

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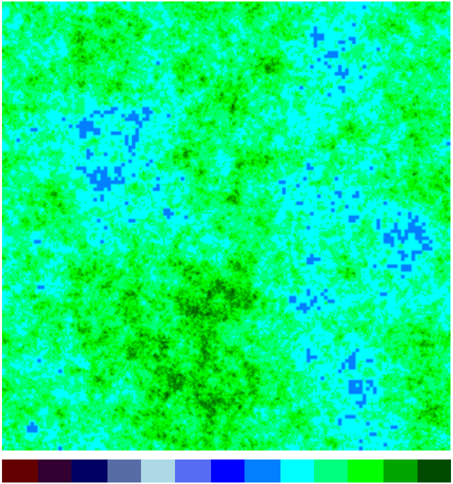

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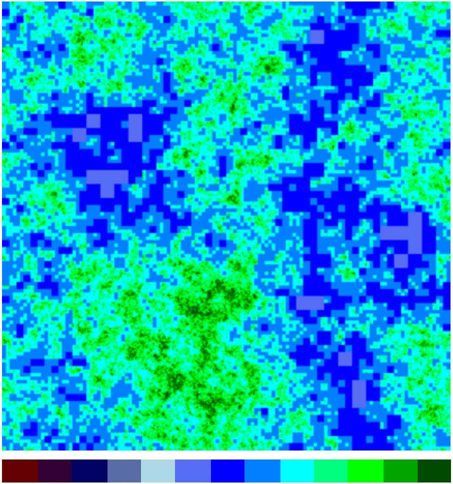

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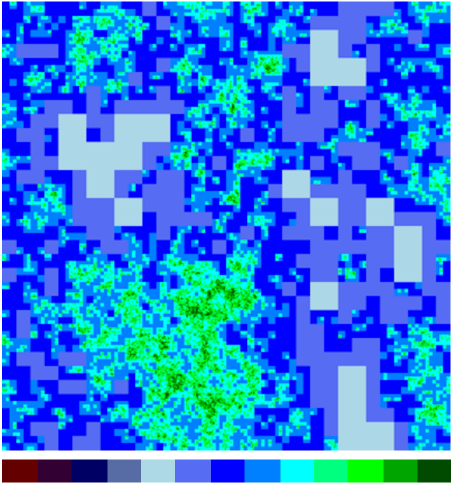

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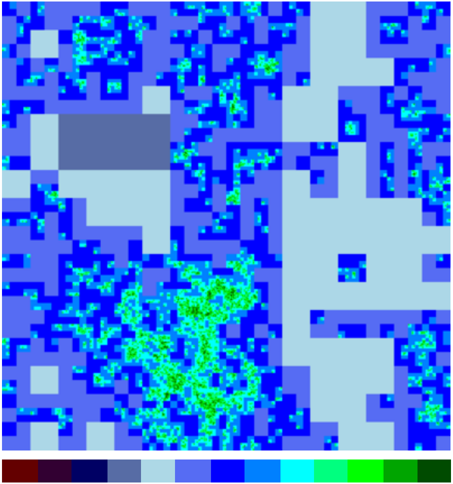

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- In contrast, TBM has a metric structure and an area measure

Canonical embedding of TBM into $\mathbf{S}^2$

- It has been believed that there should be a "natural embedding" of TBM into $\mathbf{S}^2$ and that the embedded surface is described by a form of Liouville quantum gravity (LQG) with $\gamma = \sqrt{8/3}$

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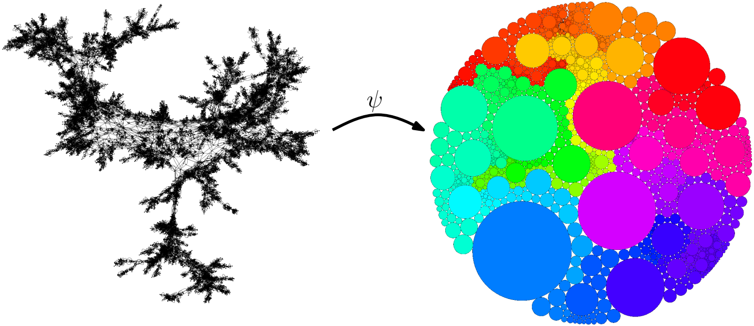

- Discrete approach: take a uniformly random planar map and embed it conformally into $\mathbf{S}^2$ (circle packing, uniformization, etc...), then in the $n \to \infty$ limit it converges to a form of $\sqrt{8/3}$-LQG.

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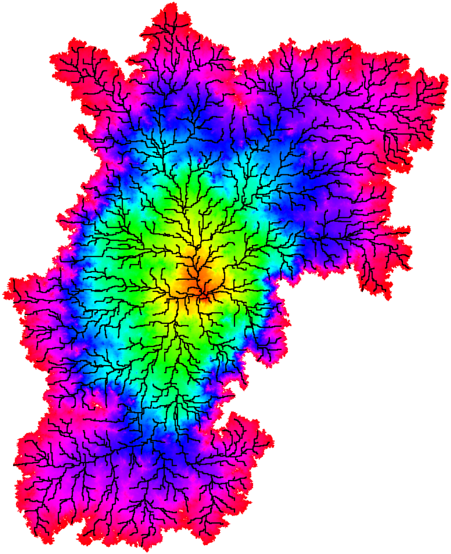
1. Construction is purely in the continuum
2. Ideas are connected to aggregation models, such as the Eden model and diffusion limited aggregation



Metric ball on a $\sqrt{8/3}$-LQG

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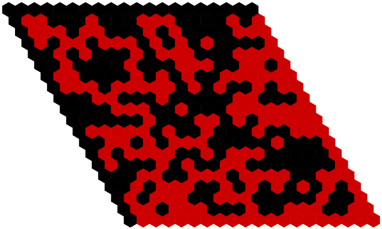
- Introduced by Schramm in '99 to describe limits of interfaces in discrete models



Critical percolation, hexagonal lattice
Each hexagon is colored red or black with prob. $\frac{1}{2}$

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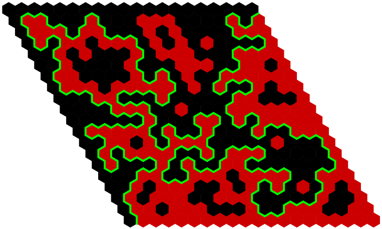

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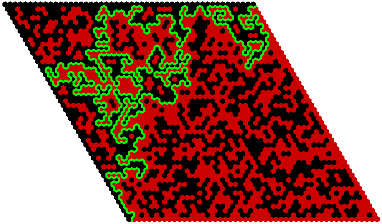

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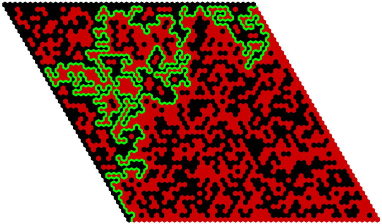

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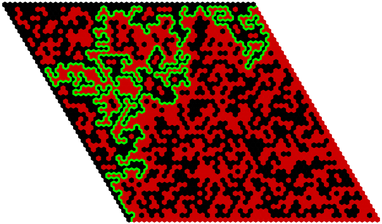

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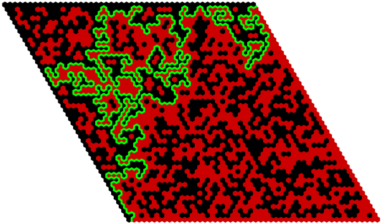

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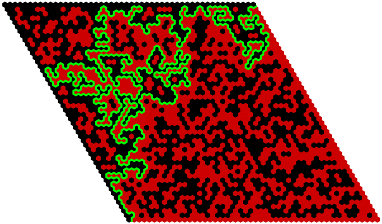

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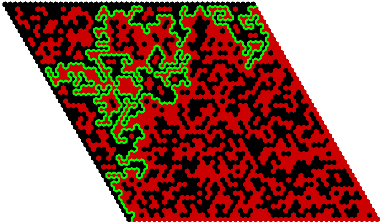
- Some special κ values:
  - $\kappa = 2$ LERW, $\kappa = 8$ UST
  - $\kappa = 8/3$ Self-avoiding walk
  - $\kappa = 3$ Ising, $\kappa = 16/3$ FK-Ising
  - $\kappa = 4$ GFF level lines
  - $\kappa = 6$ Percolation
  - $\kappa = 12$ Bipolar orientations
  - $\ldots$

(Lawler-Schramm-Werner, Smirnov, Schramm-Sheffield, ...)



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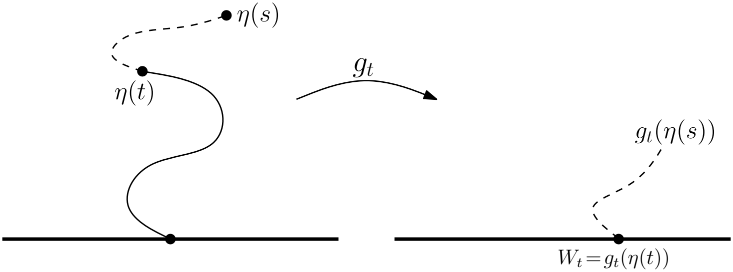

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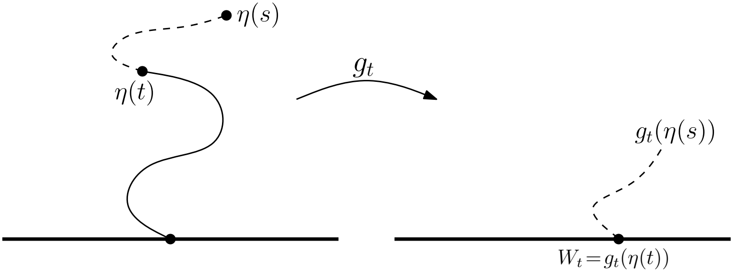

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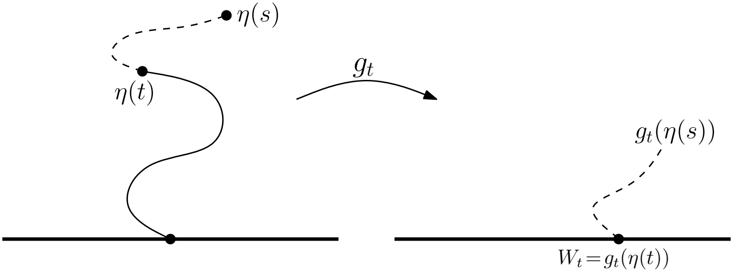

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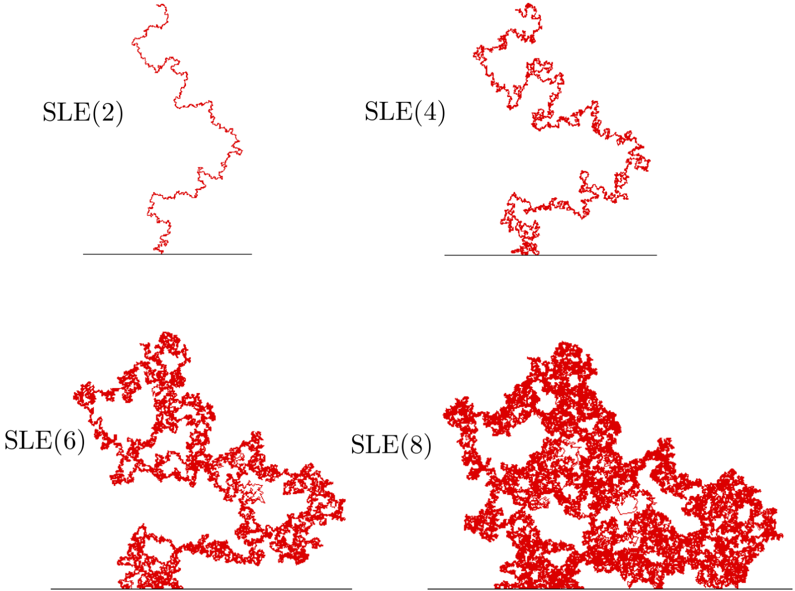

Simulations due to Tom Kennedy.

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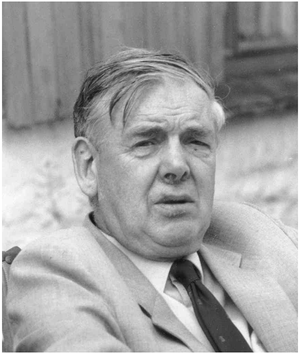

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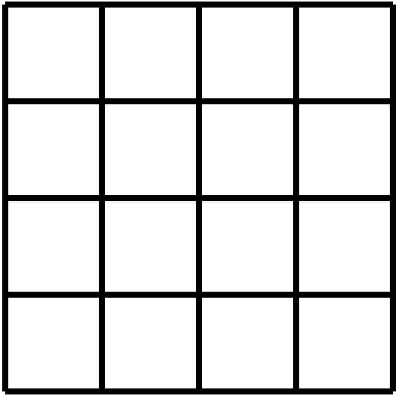

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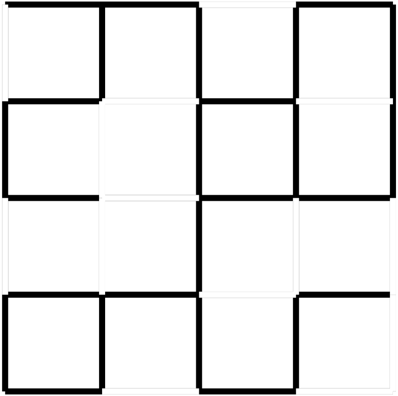

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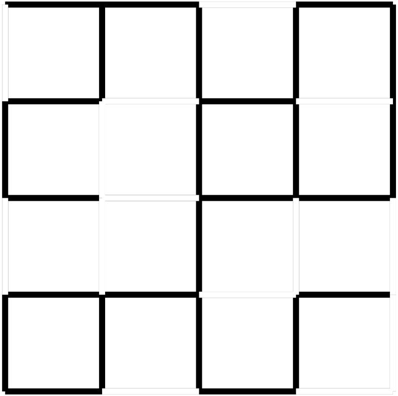

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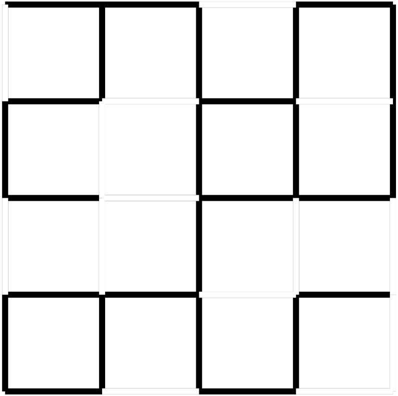

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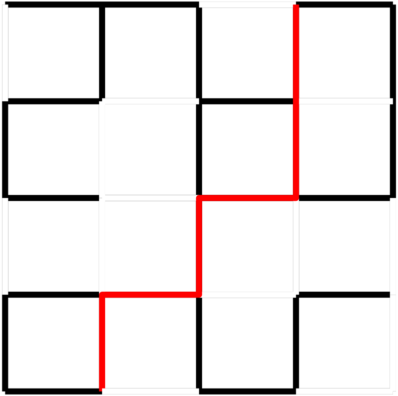

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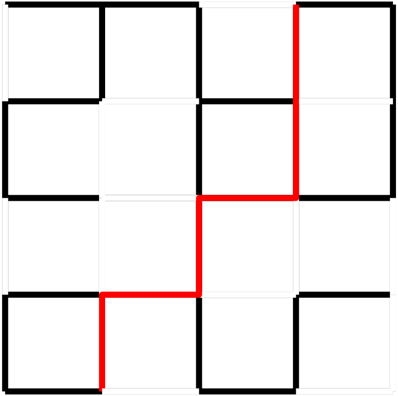
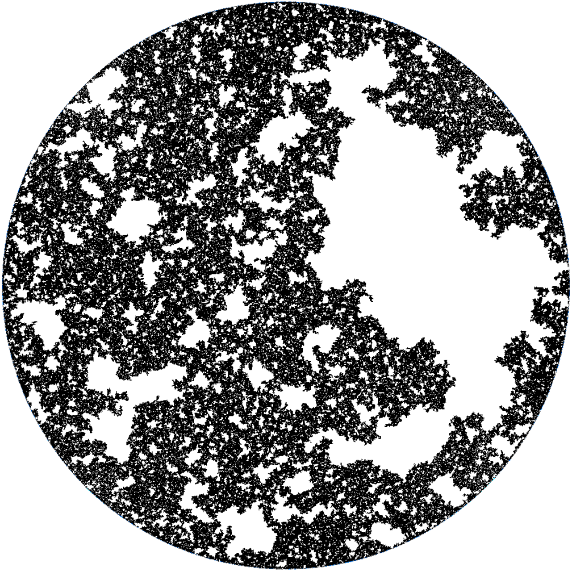
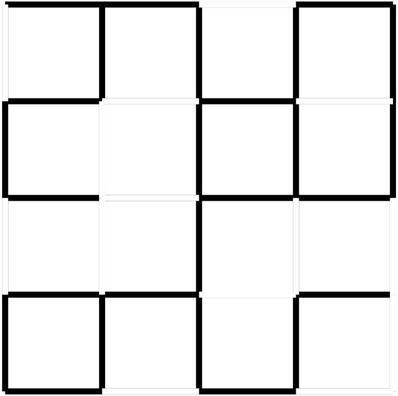




Critical bond percolation on a box in $\mathbf{Z}^2$ with side-length 1000, conformally mapped to $\mathbf{D}$. Shown are the clusters which touch the boundary.

Results on planar lattices

- $p_c = \frac{1}{2}$ for bond percolation on the $\square$-lattice



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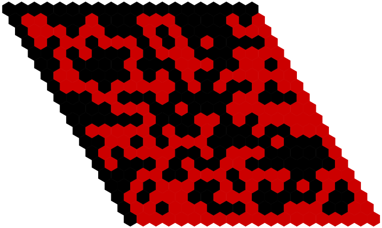

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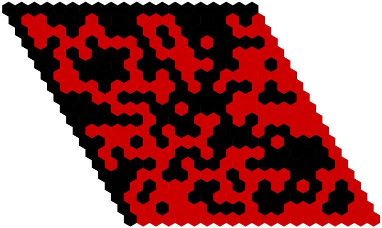

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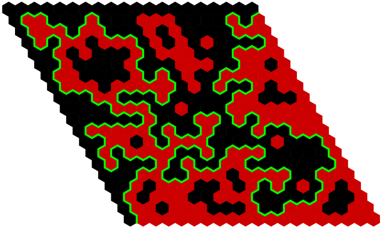

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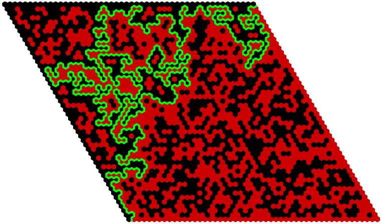

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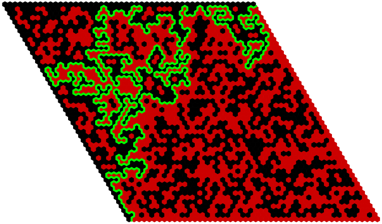

Open problem: is there any *universality*?

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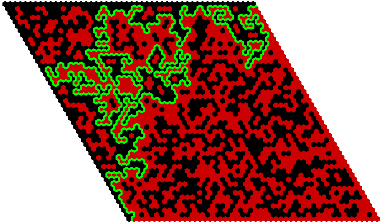

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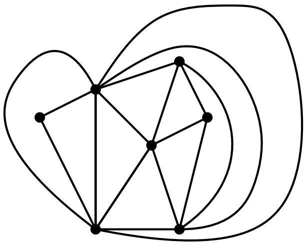
Open problem: is there any *universality*? Does the percolation exploration path converge on any other planar lattice?

Percolation on random planar maps

- **Angel:** $p_c = \frac{1}{2}$ for site percolation on a random $\triangle$



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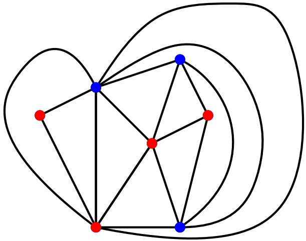

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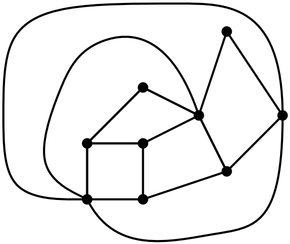

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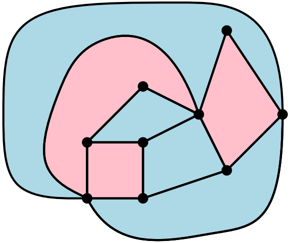

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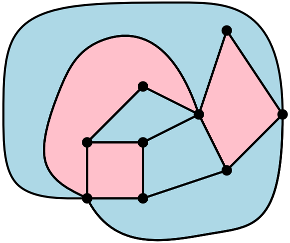

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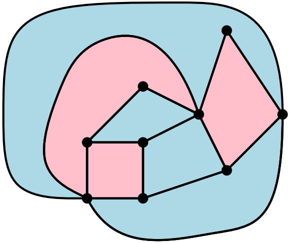

Percolation thresholds for many other types of maps have been computed (c.f. Angel-Curien, Menard-Nolin, Richlier...)

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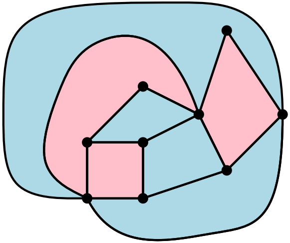

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We will consider critical $p = p_c = \frac{3}{4}$ face percolation on a random $\square$.

Percolation exploration path

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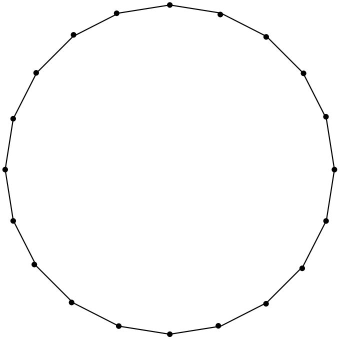

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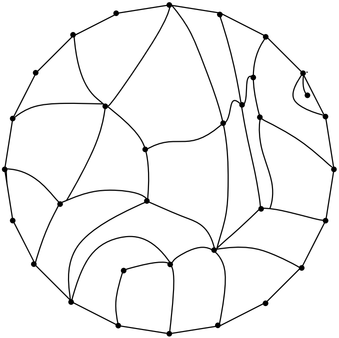
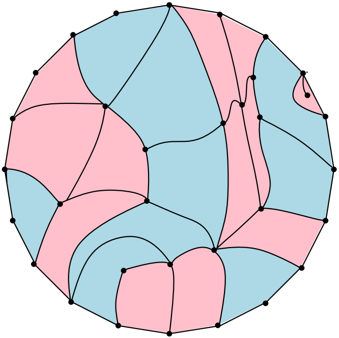


Percolation exploration path

- Work on □ of the disk
- $p = p_c = 3/4$



Percolation exploration path

- Work on □ of the disk
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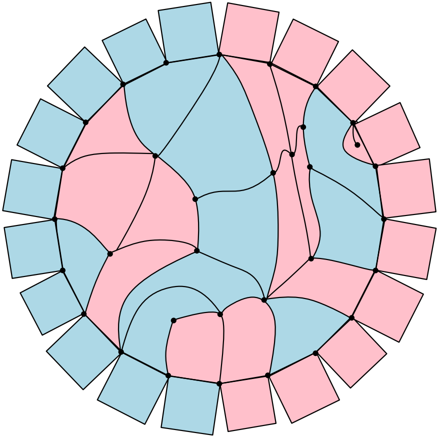

Percolation exploration path

- Work on $\square$ of the disk
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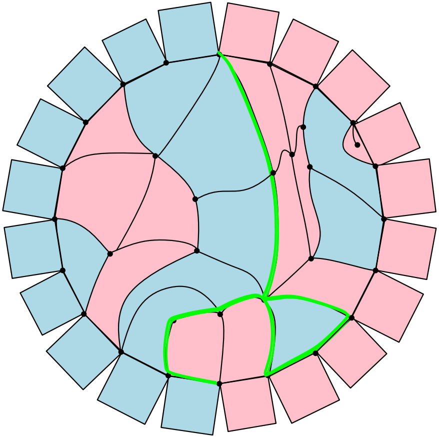
- There is a unique interface separating open/closed clusters attached to the boundary



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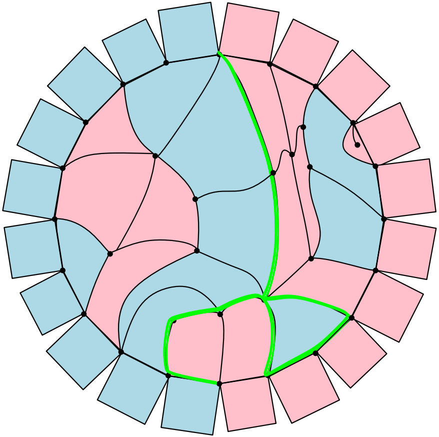

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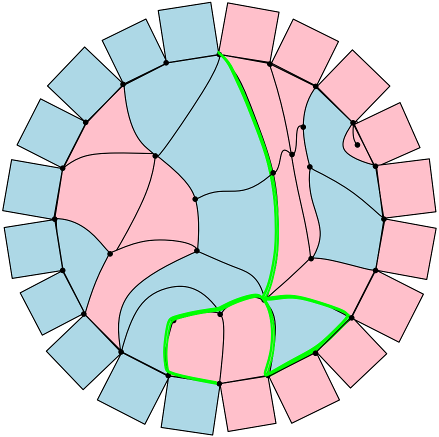

Theorem (Gwynne-M.)

The interface for critical face percolation on a random □ of the disk converges to SLE_6 *on* $\sqrt{8/3}$*-LQG.*

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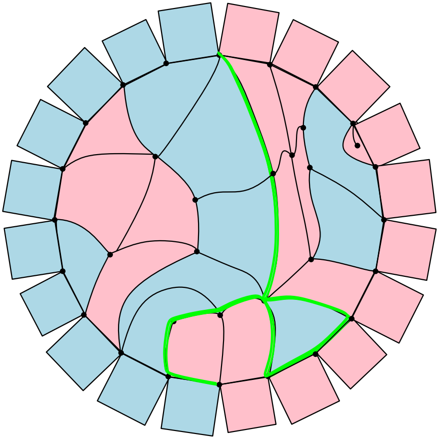

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Universal strategy: works for any random planar map model provided one has certain technical inputs.

Final words

γ-LQG: $e^{\gamma h(z)}(dx^2 + dy^2)$ where h is a GFF.

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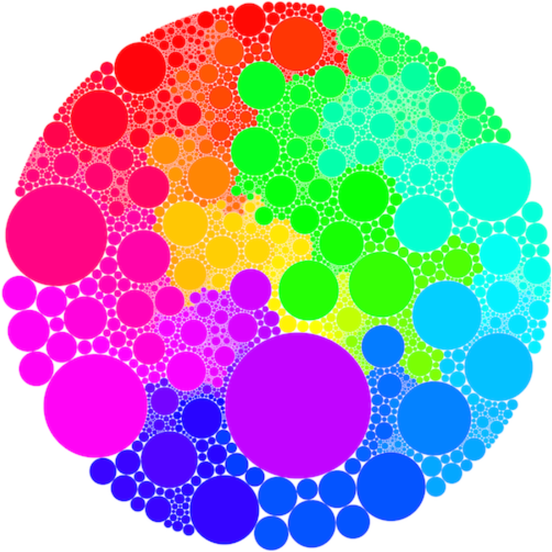
- Metric properties of γ-LQG less well-understood
- Hausdorff dimension of γ-LQG for $\gamma \neq \sqrt{8/3}$ is not known
  - Watabiki prediction:

$$d_\gamma = 1 + \frac{\gamma^2}{4} + \frac{1}{4}\sqrt{(4+\gamma^2)^2 + 16\gamma^2}.$$

  - Ding, Goswami, Gwynne, Zeitouni, Zhang.



Thanks!