Around Universality Classes in Random Matrix Theory

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Mid term Conference, Münster, March 2024

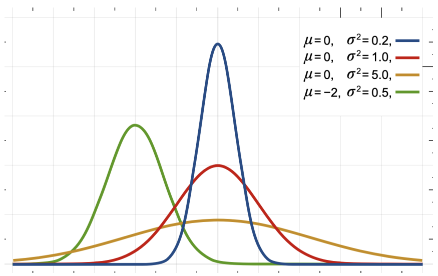
Universality Classes and Generalized CLT

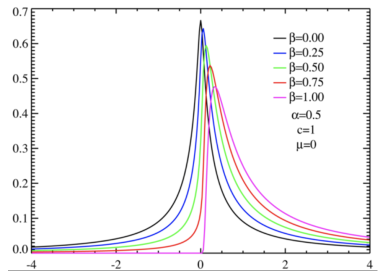
A universality class is a collection of mathematical models which share a single scale invariant limit under the process of renormalization group flow.

The simplest example is given by the Generalized Central Limit Theorem : If X_i i.i.d with law μ,

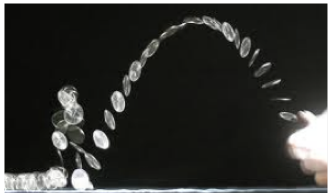
$$a_n(X_1 + \cdots + X_n - b_n n) \Rightarrow Z \quad \text{as } n \text{ goes to infinity .}$$

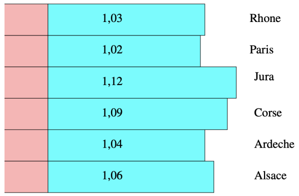
Then Z is an α-stable law. Which μ leads to a given α-stable law ?





Universality and CLT





Generalized CLT

A necessary and sufficient condition is that for all $t \in \mathbb{R}$,

$$\lim_{n\to\infty} n \log \int e^{ita_n(X-b_n)} d\mu(x) = \log \mathbb{E}[e^{itZ}]$$

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Proof Let $Z_n = a_n(X_1 + \cdots + X_n - b_n n)$. By independence, for all $t \in \mathbb{R}$,

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Remark. (1) If $\mu(x^2) < \infty$, $b_n = \mu(x)$, $a_n = n^{-1/2}$,

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(2) If $\mu(x^2) = +\infty$, typically most terms in Z_n are tiny but some are big.
(3) We can take $\mu = \mu_n$ and more generally assume X_i i.i.d law μ_n so that

$$\lim_{n\to\infty} n \log \int e^{itx} d\mu_n(x) = \log \mathbb{E}[e^{itZ}]$$

E.g. $\mu_n = \frac{p}{n}\delta_1 + (1 - \frac{p}{n})\delta_0$, $X_1^n + .. + X_n^n \Rightarrow Z$ if $\mathbb{E}[e^{itZ}] = e^{p(e^{it}-1)}$.

Wigner Random Matrices

A Wigner Random Matrix is a $n \times n$ matrix :

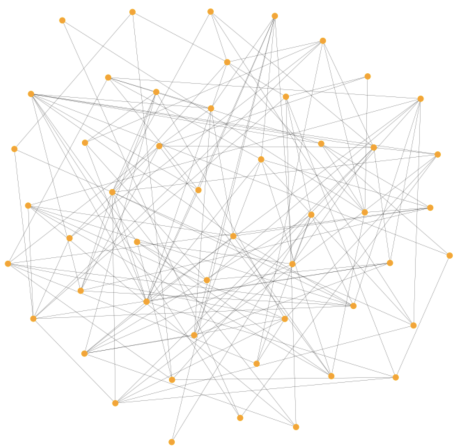
$$\mathbf{X}_n = \begin{pmatrix} x_{11} & x_{1,2} & x_{1,3} & \cdots & \cdots \\ x_{1,2} & x_{2,2} & x_{2,3} & \cdots & \cdots \\ x_{1,3} & x_{2,3} & x_{3,3} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

where
• $(x_{i,j}, 1 \le i < j \le n)$ are independent equidistributed variables with law μ_n, independent from $(x_{i,i}, 1 \le i \le n)$ independent with law ν_n.
• We assume $\mathbf{X}_n$ is symmetric $x_{ij} = x_{ji}$ for all $1 \le i, j \le n$.

Question : What is the behaviour of the spectrum and the eigenvectors of $\mathbf{X}_n$ as n goes to infinity ? How does it depend on the laws ν_n, μ_n ?

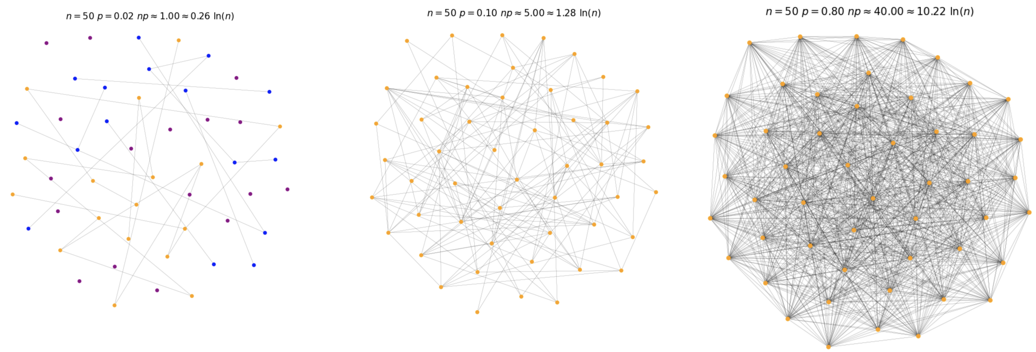
The Erdös-Rényi graph

The Erdös-Rényi graph is the graph with n vertices $\{1, \ldots, n\}$ and edges drawn at random independently with probability p.



Then, the adjacency matrix of the Erdös-Rényi graph is a Wigner matrix with $\mu_n = p\delta_1 + (1-p)\delta_0$. We call this matrix the Bernoulli matrix $\mathbf{B}_n$.

The Erdös-Rényi graph (60')



Courtesy of D. Coulette

- If $np < 1$, $G(n, p)$ has no connected component of size $> \ln n$.
- if $np \to c > 1$, $G(n, p)$ will have a unique giant connected component and lots of small components. Isolated vertices will continue to exist until $np \simeq \ln n$.
- if $np > (1 + \epsilon) \ln n$ the graph will almost surely be connected.

The spectrum of Bernoulli Random Matrix

$$\mathbf{X}_n = \frac{1}{\sqrt{p(1-p)n}}(\mathbf{B}_n - p\mathbf{1})$$

$\bar{\mathbf{X}}_n$ has independent centered entries with covariance $1/n$, and eigenvalues of order one as $\mathbb{E}[\mathrm{Tr}(\bar{\mathbf{X}}_n^2)] = \mathbb{E}[\sum \lambda_i^2] = n$. $\mathbf{1}$ is the matrix with all entries set to 1 : it has one eigenvalue equal to n, the other vanish. By Weyl's interlacing relations, their eigenvalues are close :

$$\lambda_1(\mathbf{X}_n) \leq \lambda_1(\frac{1}{\sqrt{p(1-p)n}}\mathbf{B}_n) \leq \lambda_2(\mathbf{X}_n) \leq \cdots \leq \lambda_n(\frac{\mathbf{B}_n}{\sqrt{p(1-p)n}})$$

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- Dense case $pn \gg \ln n$: The entries of $\mathbf{X}_n$ are small. We expect delocalization of the eigenvectors/continuous density. We will see $\bar{\mathbf{X}}_n$ belongs to the Gaussian matrices universality class.
- Sparse case $pn \simeq c$: most entries of $\mathbf{X}_n$ vanish, a few are of order one. We expect more localization of the eigenvectors/atoms. "$\mathbf{X}_n$ belongs to heavy tails matrices universality class"

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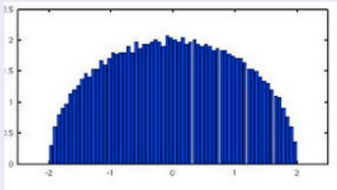
Wigner's theorem

Assume $x_{i,j} = y_{i,j}/\sqrt{n}$ with $y_{i,j}$ centered with covariance one. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $\mathbf{X}^n$.

Theorem (Wigner '56)



The empirical measure of the eigenvalues $\frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i}$ converges weakly in expectation towards the semicircle law σ : for every $a < b$,

$$\lim_{N\to\infty} \mathbb{E}[\frac{1}{n}\#\{i : \lambda_i \in [a,b]\}] = \sigma([a,b]),$$

where σ is the semicircle law :

$$\sigma(dx) = \frac{1}{2\pi}\sqrt{4-x^2}dx .$$

Heavy tails matrices

Assume that there exists $g \in L^1$ such that for all $t \in \mathbb{C}^+$

$$\lim_{n\to\infty} n(\mathbb{E}[e^{itx_{ij}^2}] - 1) = \phi(t) = \int_0^\infty g(y) e^{-i\frac{y}{t}} dy\,.$$

Theorem (Bouchaud-Cizeau '94, Zakharevich '06, Ben Arous-G '08, Benaych-Georges-G-Male '14)

The empirical measure of the eigenvalues $\frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i}$ converges weakly almost surely towards a distribution μ_ϕ given by the system

$$\begin{cases} \int \frac{1}{z-x} d\mu_\phi(x) & = i \int_0^\infty e^{iyz+\rho_z(y)} dy \\ \rho_z(\lambda) & = \lambda \int_0^\infty g(\lambda y) e^{iyz} e^{\rho_z(y)} dy \end{cases}$$

μ_ϕ has unbounded support as soon as $\limsup_{n\to\infty} \mathbb{E}[x_{ij}^2] = +\infty$.

Example : $x_{ij} = y_{ij}/n^{1/\alpha}$ with $P(|y_{ij}| \ge u) \simeq u^{-\alpha}$, $\alpha \in (0,2)$, then

$$\phi(t) = -\sigma(-it)^{\alpha/2} = \int_0^\infty C_\alpha y^{\frac{\alpha}{2}-1} e^{-i\frac{y}{t}} dy\,.$$

μ_ϕ then has a smooth density, with tail $u^{-\alpha}$.

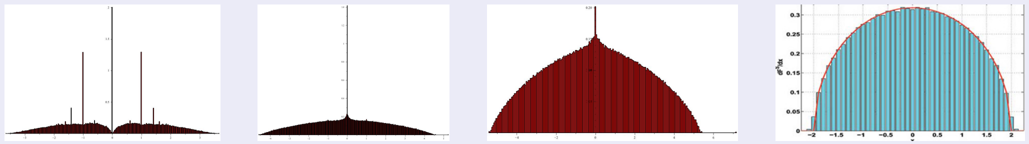
Limiting spectrum of Bernoulli Random Matrix

Assume $P(x_{ij} = 1) = 1 - P(x_{ij} = 0) = p \in (0, 1)$.

Theorem (Wigner '56 , Khorunzhy, Shcherbina, Vengerovsky '04)

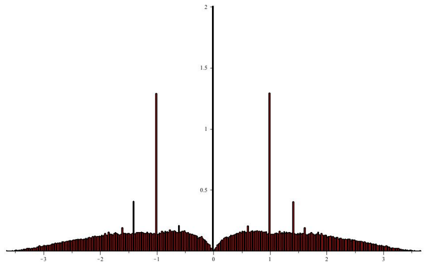
Assume pn goes to $c \in (0, +\infty]$. Then, almost surely, for any $a < b$

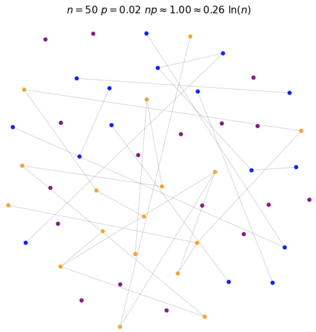
$$\lim_{n\to\infty} \frac{1}{n}\#\{i : \lambda_i \in [a, b]\} = \mu_c([a, b])$$



Simulation for $c = 1, 2, 3, \infty$ (Courtesy of J. Salez)

- When $c = \infty$, $\mu_\infty(dx) = \frac{1}{2\pi}\sqrt{4 - x^2}dx$ is the semi-circle law.
- μ_c has a continuous part iff $c > 1$ (Bordenave, Sen, Virag '17) and atoms at all totally real algebraic integers for all $c \in (0, \infty)$ (Salez '15)





Atoms are for instance created by small connected components

Idea of the proof

Enough to show the convergence of the Stieljes transform given for $z \in \mathbb{C}\backslash\mathbb{R}$ by

$$G_n(z) = \frac{1}{n}\mathrm{Tr}(z - \mathbf{X}_n)^{-1} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{z - \lambda_i}\,.$$

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If $X_i = (x_{ji})_{j \neq i}$ and $\mathbf{X}_n^{(i)}$ its principal minor, Schur complement formula implies :

$$(z - \mathbf{X}_n)^{-1}_{ii} = \frac{1}{z - X_{ii} - \langle X_i, (z - \mathbf{X}_n^{(i)})^{-1} X_i \rangle}$$

where
$\langle X_i, (z - \mathbf{X}_n^{(i)})^{-1} X_i \rangle = \sum_{j,k \neq i} x_{ji} x_{ki} (z - \mathbf{X}_n^{(i)})^{-1}_{jk} \simeq \sum_{j \neq i} x_{ji}^2 (z - \mathbf{X}_n^{(i)})^{-1}_{jj}$. Then

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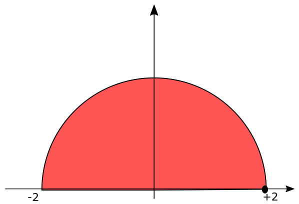
- If $\mathbb{E}[n x_{ij}^2] = +\infty$, $\sum_{j \neq i} x_{ji}^2 (z - \mathbf{X}_n^{(i)})^{-1}_{jj}$ stays random. But, the Fourier transform of this random variable is solution of an equation with a unique solution.

Extreme eigenvalues



Theorem (Soshnikov '04, Auffinger, Ben Arous, Péché ' 09, Lee-Yin '14, Benaych-Georges, Bordenave, Knowles '19,Alt, Ducatez, Knowles' 19, Hiesmayr,McKenzie '23)

- *The eigenvalues stick to the bulk ($\lambda_n \to 2$ a.s.) iff* $\mathbb{E}[(\sqrt{n}x_{ij})^4] < \infty$.
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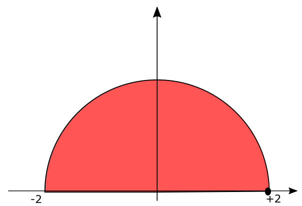

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  - *If* $pn \in [(\ln n)^{-1/10}, \ln n/(\ln 4 - 1))$, *then if* α_n^* *is the largest degree and* β_n^* *the number of vertices at distance 2 from the vertex with this degree,*

$$\lambda_n \simeq f(\alpha_n^*, \beta_n^*) \quad (\simeq \sqrt{\ln n/\ln\ln n} \text{ if } pn/\ln n \to 0.)$$

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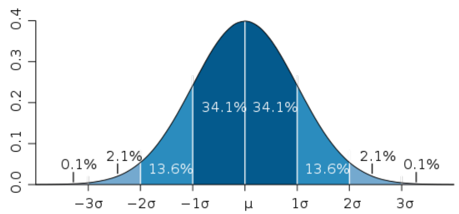
Global Fluctuations and central limit theorem



Theorem (Jonsson '82, Johansson '98, Shcherbina-Tirozzi '10, Benaych-Georges-G-Male '14, He '20)

Take $f \in C_b^1$. Then,

$$\sqrt{n}^{-h} \sum_{i=1}^{n} (f(\lambda_i) - \mathbb{E}[f(\lambda_i)]) \to N(0, \sigma_\phi(f))$$

where $h = 1$ for heavy tails and $h = 0$ for light tails $\sup_n \max_{ij} \mu((\sqrt{n} x_{ij})^4) < \infty$. For Bernoulli matrices $\sqrt{n}^{-h}$ is replaced by $\sqrt{p}$.

Local fluctuations, the Gaussian case

Take $\mathbf{G}_n$ a GOE matrix, i.e $n \times n$ symmetric matrix with independent centered Gaussian entries with covariance $1/n$. Then for any orthogonal matrix $\mathbf{O}$,

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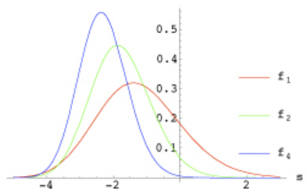

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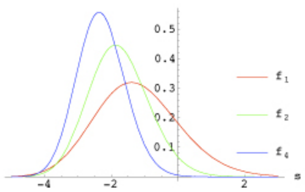

- The fluctuations of the eigenvalues in the bulk can be analyzed, see e.g. Mehta, to get the vague convergence of $n(\lambda_i - E), 1 \leq i \leq n$.

Gaussian Universality Class

Wigner matrices with light tails

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Theorem (Erdos-Schlein-Yau et al 11-15, Tao-Vu '11, Lee-Yin '14)

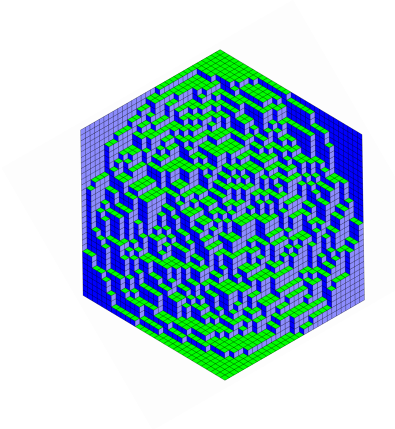
- *If $\mathbb{E}[|\sqrt{n}x_{ij}|^4] < \infty$, the fluctuations of the largest eigenvalues are the same as in the Gaussian case,*
- *If $E[|\sqrt{n}x_{ij}|^2] < \infty$, the fluctuations of the eigenvalues in the bulk are the same than in the Gaussian case.*

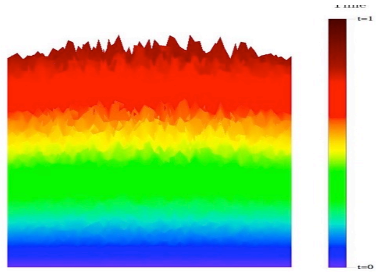
Bernoulli matrices in the dense case

Theorem (Erdös, Knowles, Yau, Yin '11, Huang, Landon, Yau '15, Lee,Schnelli '18, see also Tao-Vu et al '10)

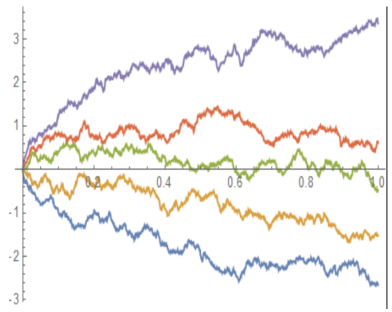
Assume $pn \gg n^{\varepsilon}$. Then the local fluctuations of the spectrum and the delocalization of the eigenvectors of $\mathbf{B}_n$ are the same as those of the Gaussian ensemble $\mathbf{G}_n$ in the bulk for $\varepsilon > 0$, for the largest eigenvalue if $\varepsilon > 1/3$.

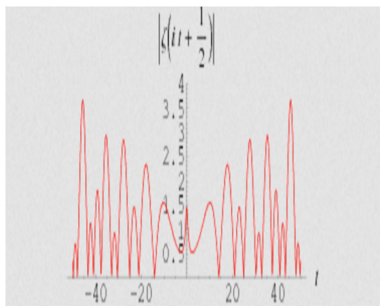
More Universality





The evolution of a stochastic interface at different times (indicated through different colors). Figure by Alexandre Krajenbrink.





Other Universality Classes

Theorem (Largest Eigenvalues)

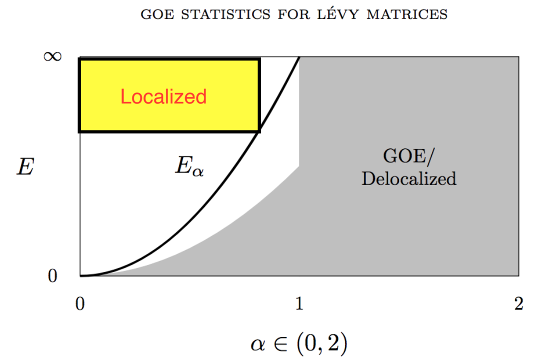
- *Auffinger, Ben Arous, Péché '07 : Assume $P(|x_{ij}| \geq t) \simeq t^{-\alpha}$, $\alpha \in (0,4)$. Then, $\sum \delta_{n^{-\frac{2}{\alpha}}\lambda_i}$ converges towards a Poisson process with intensity $\alpha x^{-1-\alpha}$.*
- *Alt, Ducatez and Knowles '21, Hiesmayr,McKenzie '23 : When $pn \in [(\ln n)^{-1/10}, C \ln n]$, the largest eigenvalues of a Bernoulli matrix are asymptotically distributed like a Poisson process and their eigenvectors are localized (around vertices with high degree).*

Very little is known about the fluctuations in the bulk.

Local fluctuations in the bulk

When the entries are α-stable : $\mathbb{P}(|A_{ij}| \geq t) \simeq t^{-\alpha}/n$

- Local laws are obtained and a transition is shown (Bordenave-G '13, '17)
- In a large region the local fluctuations in the bulk are like Gaussian (Aggarwal, Lopatto, Yau '18).
- The transition should happen at the mobility edge (Tarquini, Biroli, Tarzia '16/ Aggarwal, Bordenave, Lopatto '22) which is characterized by a transition delocalized/localized eigenvectors.



Ideas of the Proof

- When the entries are Gaussian, the joint law of the eigenvalues is explicit :

$$d\mathbb{P}(\lambda) = \frac{1}{Z_n} \prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^\beta e^{-\frac{n}{4}\sum_i \lambda_i^2} \prod d\lambda_i$$

  Based on this formula, local fluctuations can be derived.
- A local law can be proven in the spirit of Erdos-Schlein-Yau ' 11 : if $b - a \gg 1/n$

$$P(|\frac{1}{n}\sum 1_{\lambda_i \in [a,b]} - \sigma([a,b])| > \delta |b-a|) \le Ce^{-c\delta\sqrt{n|b-a|}}$$

- Fluctuations can be compared to the Gaussian case in Lindenberg spirit (Tao and Vu) or by using the Gaussian semi-group (Erdös-Yau et al).
- Much more complicated in the heavy tails case because the system of equations describing the limit law is more involved.

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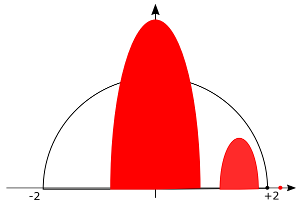
Rare events, the Gaussian case



The joint law of the eigenvalues of a GOE matrix is given by

$$d\mathbb{P}_n(\lambda) = \frac{1}{Z_n} \prod_{1 \le i < j \le n} |\lambda_i - \lambda_j| e^{-\frac{n}{4}\sum_i \lambda_i^2} \prod d\lambda_i$$

Theorem

- *(Ben Arous-G '97) For any probability measure μ on $\mathbb{R}$*

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i} \simeq \mu\right) \simeq e^{-n^2(J(\mu) - \inf J)}$$

with $J(\mu) = \frac{1}{4}\int x^2 d\mu(x) - \frac{1}{2}\int\int \ln|x-y| d\mu(x) d\mu(y)$.

- *(Ben Arous, Dembo, G. '01) For any $x \geq 2$,*

$$\mathbb{P}(\lambda_n \simeq x) \simeq e^{-n\int_2^x \sqrt{4-y^2}dy}.$$

Rare events, very heavy tails

Theorem

- *(Bordenave-Caputo '15, Bordenave-G-Male WIP) For Bernoulli matrices with $np \simeq c \in (0, +\infty)$ or α-stable entries ($\alpha < 4$),*

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i} \simeq \mu\right) \simeq e^{-n(J_\Phi(\mu) - \inf J_\Phi)}$$

Rare events, very heavy tails

Theorem

- *(Bordenave-Caputo '15, Bordenave-G-Male WIP) For Bernoulli matrices with $np \simeq c \in (0, +\infty)$ or α-stable entries ($\alpha < 4$),*

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i} \simeq \mu\right) \simeq e^{-n(J_\Phi(\mu) - \inf J_\Phi)}$$

- *(Augeri, Basak '23) Let $x_{ij} = b_{ij}g_{ij}/\sqrt{np}$ with $P(b_{ij} = 1) = p$, $\ln n \ll pn \ll n$, g_{ij} $N(0,1)$.*

$$\mathbb{P}(\lambda_n \simeq x) \simeq e^{-npJ(x)}$$

- *(Bhattacharya, Bhattacharya, Ganguly '20). Bernoulli matrices : Assume $\frac{1}{n} \ll pn \ll \sqrt{\ln n/\ln\ln n}$. Then, with $x_n \simeq \sqrt{\ln n/\ln\ln n}$,*

$$P(\lambda_n \geq (1+\delta)x_n) \simeq n^{-(2\delta+\delta^2)}.$$

Large deviations are generated by either the emergence of a high degree vertex with a large vertex weight or that of a clique with large edge weights.

Rare events, tails heavier than Gaussians

Assume that for some $\alpha \in (0,2)$, there exists $a > 0$ so that for all i, j

$$\lim_{t \to \infty} 2^{-1_{i=j}} t^{-\alpha} \ln \mathbb{P}(|\sqrt{n} x_{ij}| \geq t) = -a$$

Theorem

- *(Bordenave-Caputo '12) The law of the empirical measure satisfy a LDP in the speed* $n^{1+\frac{\alpha}{2}}$ *and good rate function which is infinite unless* $\mu = \sigma \boxplus \nu$ *and then equals* $a \int |x|^\alpha d\nu(x)$.
- *(Augeri '15) The law of the largest eigenvalue satisfies a LDP with rate* $n^{\frac{\alpha}{2}}$ *and GRF equals* $c(\int (x-y)^{-1} d\sigma(y))^{-\alpha}$.

Hint : To create an atypical behaviour of the largest eigenvalue (resp. empirical measure), it is enough to create one (resp$\simeq n$) big entry, with probability of order $e^{-O(n^{\frac{\alpha}{2}})}$ (resp. $(e^{-O(n^{\frac{\alpha}{2}})})^n$) .

Universality of large deviations for sharp sub-Gaussian entries

μ has a sub-Gaussian tail if there exists $A \geq 1$ such that for all t

$$\int e^{tx} d\mu(x) \leq e^{A\frac{t^2}{2}\mu(x^2)}.$$

μ has a sharp subgaussian tail iff $A = 1$. The Rademacher law $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}$ and the uniform measure $[-\sqrt{3}, \sqrt{3}]$ have sharp sub-gaussian tails.

Theorem (G-Husson '18)

Assume the entries x_{ij} have a sharp sub-Gaussian tail (+ compact support or log-Sobolev inequality). Then the law of λ_1 satisfies the same large deviation principle than in the Gaussian case : for all x

$$\mathbb{P}(\lambda_1 \simeq x) \simeq e^{-nI_{GOE}(x)}$$

Large deviations in the sub-Gaussian case

Take $x_{ij} = y_{ij}/\sqrt{n}$, y_{ij} with law μ. Assume μ is symmetric and sub-Gaussian :

$$A := \sup_{t\in\mathbb{R}} \frac{2}{t^2\mu(x^2)} \ln \int e^{tx} d\mu(x) \in [1, +\infty) .$$

Theorem (Augeri-G-Husson '19, Cook-Ducatez-G '23)

Assume $A > 1$. Under some technical hypothesis, the law of λ_1 satisfies large deviation principle with good rate function I_μ : for every real number x

$$\mathbb{P}(\lambda_1 \simeq x) \simeq e^{-nI_\mu(x)} ,$$

where $I_\mu(x) \simeq \frac{x^2}{4A} < I_{GOE}(x)$ for x large and $I_\mu(x) = I_{GOE}(x)$ for x small.

The universal region corresponds to delocalized eigenvectors conditionally to the deviation, whereas its complement corresponds to localized eigenvectors (on one to $\sqrt{n}$ sites).

Some ideas

- Use explicit formulas when they exists (Gaussian case),
- When the tail is sufficiently heavy, create deviations by localized changes,
- In other cases, use Fourier analysis, namely spherical integrals and ideas from Cramer's proof of large deviations.

A message and some open problems

Universality classes are related to a localization phenomenon.

It is not yet understood in many cases :

- Local fluctuations in the bulk for most heavy tail matrices,
  - Local law : Only known for α stable laws. What about Bernoulli ? Difficulty related with presence of atoms, and technically on complexity of the equations describing the limiting laws,
  - Mobility edge : When do we observe a transition in local behaviour inside the bulk ?
  - Limit laws : What laws describe the local fluctuations ? (Poisson ?)
- Large deviations for the empirical measure for sub-Gaussian entries.

Extensions of many (but not all) of these results exist for other matrix models (Wishart, Structured matrices, tensors etc).

Thank You for your Attention.