Cohomology of moduli spaces: a case study

Oscar Randal-Williams

European Research Council
Established by the European Commission

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$$\mathcal{M}_g^{trop} = \{\text{space of metric graphs with } g \text{ loops}\}$$
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Mumford '67: $H^1(\mathcal{M}_g; \mathbb{Z}) = 0$.

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Its breakthrough result was the resolution of Mumford's conjecture on the cohomology of Riemann's moduli space:

Madsen–Weiss '07: $H^*(\mathcal{M}_g; \mathbb{Q}) = \mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ for $* \leq \frac{2g-2}{3}$.

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This new community has its own point of view and toolbox, inspired by Quillen's foundations for algebraic K-theory and many other developments in Homotopy Theory.

In situations where this point of view applies it has often led to significant results.

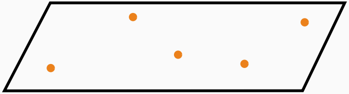
A case study

Points in the plane

Moduli space of n distinct unordered particles in the complex plane:

$$\mathrm{Conf}_n(\mathbb{C}) = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}/\mathfrak{S}_n$$

$$= \{d \in \mathbb{C}[t] \mid {}^{\text{degree } n \text{ monic polynomial}}_{\text{with no repeated roots}}\}$$



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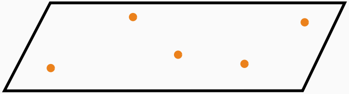
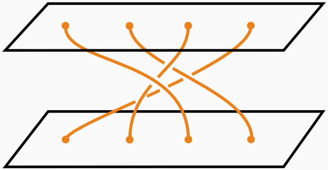


$$\pi_1(\mathrm{Conf}_n(\mathbb{C})) = \beta_n$$

Artin's braid group on n strands



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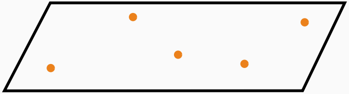

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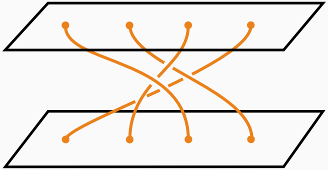

Arnold '70: $H^i(\mathrm{Conf}_n(\mathbb{C}); \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0, 1 \\ 0 & \text{else} \end{cases}$ for all $n \geq 2$.

Branched covers

Associated to a degree n monic polynomial d with no repeated roots is the smooth Riemann surface

$$C_d = \{(x, y) \in \mathbb{C}^2 \mid y^2 = d(x)\},$$

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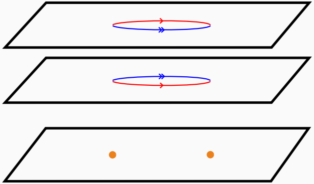
It is a genus $\frac{n-1}{2}$ surface with a point removed if n is odd, and a genus $\frac{n-2}{2}$ surface with two points removed if n is even.



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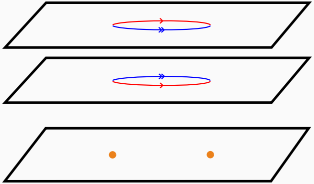

The assignment $d \mapsto H^1(C_d; \mathbb{Z})$ defines a local system $\mathbb{V}$ of symplectic forms over $\mathrm{Conf}_n(\mathbb{C})$, so a symplectic representation of the braid group β_n: this is "the reduced integral Burau representation".

The question

For simplicity suppose from now on that $n = 2g + 1$ is odd, so the corresponding branched cover is a genus g surface with a point removed.

Then "the reduced integral Burau representation" has the form

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More generally, the irreducible representations of Sp_{2g} are named $\mathbb{V}_\lambda$ for partitions λ of length $\leq g$, and one may ask:

Question: What is $H^*(\mathrm{Conf}_{2g+1}(\mathbb{C}); \mathbb{V}_\lambda) = H^*(\beta_{2g+1}; \mathbb{V}_\lambda)$?

Motivation for the question

Let q be an odd prime power. For $d \in \mathbb{F}_q[t]$ monic and squarefree of degree $n = 2g + 1$ there is a curve

$$C_d = \{(x, y) \in \mathbb{A}^2_{\mathbb{F}_q} \mid y^2 = d(x)\}.$$

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We may count the number of solutions of the equation $y^2 = d(x)$ over each finite extension $\mathbb{F}_{q^k}$ of $\mathbb{F}_q$. Then

$$\exp\left(\sum_{k\geq 1} \#C_d(\mathbb{F}_{q^k})\frac{t^k}{k}\right) = \frac{P_{C_d}(t)}{(1-t)(1-qt)} \text{ for all } |t| < \tfrac{1}{q}$$

for an integer polynomial P_{C_d}.

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Conjecture$^{(*)}$ of Conrey–Farmer–Keating–Rubinstein–Snaith

For fixed q and r

$$q^{-2g-1} \sum_{\substack{d \text{ monic, squarefree} \\ \text{of degree } 2g+1}} P_{C_d}(q^{-1/2})^r = Q_r(2g+1) + o(1) \quad \text{as } g \to \infty$$

for an explicit polynomial Q_r of degree $r(r+1)/2$.

Arithmetic and topology

For suitable schemes X over $\mathbb{Z}$ there is an incredible relationship between arithmetic and topology, specifically between

$$\text{(weighted) counts of } X(\mathbb{F}_{q^k}) \quad \text{and} \quad \text{(twisted) cohomology of } X(\mathbb{C})$$

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Example of the principle

$$\#X(\mathbb{F}_q) = q^{\dim(X)} \sum_i (-1)^i \mathrm{Tr}(\mathrm{Frob}_q : H_i^{\text{ét}}(X_{\bar{\mathbb{F}}_q}) \to H_i^{\text{ét}}(X_{\bar{\mathbb{F}}_q}))$$

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Applied with $X = \mathrm{Conf}_n(\mathbb{A}^1)$ and $H^*(\mathrm{Conf}_n(\mathbb{C}); \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0, 1 \\ 0 & \text{else} \end{cases}$

$$\Rightarrow \qquad \#\{d \in \mathbb{F}_q[t] \mid \substack{\text{degree } n \text{ monic polynomial} \\ \text{with no repeated roots}}\} = q^n(1 - q^{-1})$$

Work of Bergström–Diaconu–Petersen–Westerland

Bergström–Diaconu–Petersen–Westerland ’23 apply this to the scheme $\mathrm{Conf}_{2g+1}(\mathbb{A}^1)/\mathbb{G}_a$ and the local system $(\Lambda^\bullet \mathbb{V})^{\otimes r}$.

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(i) completely calculate $\lim_{g\to\infty} H^*(\mathrm{Conf}_{2g+1}(\mathbb{C}); \mathbb{V}_\lambda)$, showing that with the $p_{\lambda,r}$ it recovers the right-hand side in the CFKRS conjecture, and

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(ii) explain how that conjecture would follow (for fixed r and all large enough q) if

$$H^i(\mathrm{Conf}_{2g+1}(\mathbb{C}); \mathbb{V}_\lambda) \xleftarrow{\sim} H^i(\mathrm{Conf}_{2(g+1)+1}(\mathbb{C}); \mathbb{V}_\lambda)$$

for all $i \leq A \cdot g - B$, some $A > 0$.

Strategy of Bergström–Diaconu–Petersen–Westerland

Modern homotopical methods for calculating stable cohomology of e.g. $\mathrm{Conf}_n(\mathbb{C})$ exploit *locality*:

$$U \longmapsto \{\text{discrete subsets of } U\}$$

is a sheaf of spaces on $\mathbb{C}$ (and almost a homotopy sheaf).

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However: for a topological space X,

$$\mathrm{Conf}_n(\mathbb{C})^X := \{(d, \varphi) \mid d \in \mathrm{Conf}_n(\mathbb{C}), \varphi : C_d \to X\}$$

is describable locally on $\mathbb{C}$.

The homotopical methods apply here, functorially in X. This gives a large (enough) supply of local systems on $\mathrm{Conf}_n(\mathbb{C})$ for which one can calculate the stable cohomology.

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Bergström, Diaconu, Petersen, and Westerland reduce the CFKRS conjecture to showing: there are $A > 0$, B such that

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Showing this for $i \leq g - \frac{1+|\lambda|}{2}$ is by now routine in the subject of homological stability (it follows from **R-W–Wahl '17**) but is useless here: BDPW need a single stability range that works for all λ at once.

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Miller–Patzt–Petersen–R-W '24: It holds for all $i \leq \frac{1}{6} \cdot g - 1$.

$$\Rightarrow \quad q^{-2g-1} \sum_{\substack{d \text{ monic, squarefree} \\ \text{of degree } 2g+1}} P_{C_d}(q^{-1/2})^r = Q_r(2g+1) + O(4^{g(r+1)} q^{-(g+6)/12})$$

$\Rightarrow$ CFKRS conjecture for all $q > 2^{24(r+1)}$

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We then show that $C_*(\Gamma_n; \mathbb{V}_\lambda)$ can be constructed in a precisely controlled way from the collection

$$\{C_*(\beta_m; \mathbb{V}_\mu)\}_{m \leq n, \mu \leq \lambda},$$

in such a way that if our required stability theorem did not hold, then neither could Borel's.

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