Resonances as a computational tool

Katharina Schratz
Laboratoire Jacques-Louis Lions
Sorbonne Université

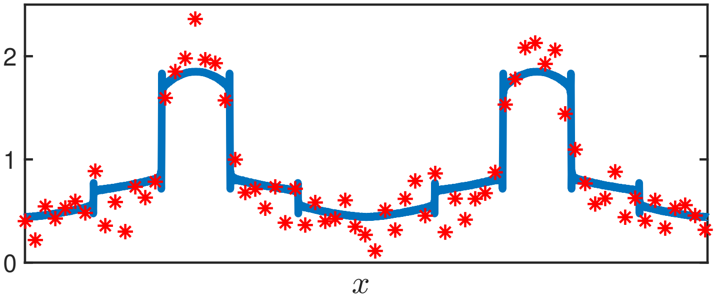
Mathematics Münster Mid-term Conference
27 Mars 2024

European Research Council
Established by the European Commission

SORBONNE UNIVERSITÉ

Numerics for non-smooth phenomena
in nonlinear dispersive PDEs (and beyond) ?

Talbot effect (dispersive quantisation)



blue: exact solution, red: numerical (Strang splitting)

Model problem: Korteweg–de Vries equation

$$\partial_t u(t,x) + \partial_x^3 u(t,x) + \frac{1}{2}\partial_x u^2(t,x) = 0$$

Model problem: Korteweg–de Vries equation

$$\partial_t u(t,x) + \partial_x^3 u(t,x) + \frac{1}{2}\partial_x u^2(t,x) = 0$$

◦ Splitting methods:

$$\text{(S1)} \quad \partial_t u + \partial_x^3 u = 0 \qquad \text{(S2)} \quad \partial_t u + \frac{1}{2}\partial_x u^2 = 0$$

Model problem: Korteweg–de Vries equation

$$\partial_t u(t,x) + \partial_x^3 u(t,x) + \frac{1}{2}\partial_x u^2(t,x) = 0$$

◦ Splitting methods:

$$\text{(S1)} \quad \partial_t u + \partial_x^3 u = 0 \qquad \text{(S2)} \quad \partial_t u + \frac{1}{2}\partial_x u^2 = 0$$

Numerical approximation: $u(0) \mapsto u^1$ (Strang)

$$u^1 = \varphi_{\mathsf{S1}}^{\tau/2} \circ \varphi_{\mathsf{S2}}^{\tau} \circ \varphi_{\mathsf{S1}}^{\tau/2}(u(0))$$

Model problem: Korteweg–de Vries equation

$$\partial_t u(t,x) + \partial_x^3 u(t,x) + \frac{1}{2}\partial_x u^2(t,x) = 0$$

◦ Splitting methods:

$$\text{(S1)} \quad \partial_t u + \partial_x^3 u = 0 \qquad \text{(S2)} \quad \partial_t u + \frac{1}{2}\partial_x u^2 = 0$$

Numerical approximation: $u(0) \mapsto u^1$ (Strang)

$$u^1 = \varphi_{\mathsf{S1}}^{\tau/2} \circ \varphi_{\mathsf{S2}}^{\tau} \circ \varphi_{\mathsf{S1}}^{\tau/2}(u(0))$$

Theorem (global error): [Holden, Karlsen, Risebro, Tao '12]

$$\|u(t_n) - u^n\|_{H^1} \le c(t_n)\tau^2 \|u\|_{L^\infty_{t_n} H^6}$$

Assumptions:

- ◦ Burger eq is solved exactly
- ◦ smooth solutions, e.g., $u \in L^\infty_{t_n} H^6$

The Korteweg–de Vries equation

$$\partial_t u(t,x) + \partial_x^3 u(t,x) + \frac{1}{2}\partial_x u^2(t,x) = 0$$

◦ Exponential integrators:

$$u(t) = \mathrm{e}^{-t\partial_x^3} u(0) - \frac{1}{2}\mathrm{e}^{-t\partial_x^3}\partial_x \int_0^t \mathrm{e}^{s\partial_x^3} u^2(s)\mathrm{d}s$$

The Korteweg–de Vries equation

$$\partial_t u(t,x) + \partial_x^3 u(t,x) + \frac{1}{2}\partial_x u^2(t,x) = 0$$

- Exponential integrators:

$$\begin{aligned} u(t) &= \mathrm{e}^{-t\partial_x^3} u(0) - \frac{1}{2}\mathrm{e}^{-t\partial_x^3}\partial_x \int_0^t \mathrm{e}^{s\partial_x^3} u^2(s)\mathrm{d}s \\ &= \mathrm{e}^{-t\partial_x^3} u(0) - \frac{1}{2}\mathrm{e}^{-t\partial_x^3}\partial_x \int_0^t \mathrm{e}^{s\partial_x^3}\mathrm{d}s \; \left(u^2(0)\right) + \mathcal{O}\left(t^2 \partial_x \partial_t u\right) \end{aligned}$$

The Korteweg–de Vries equation

$$\partial_t u(t,x) + \partial_x^3 u(t,x) + \frac{1}{2}\partial_x u^2(t,x) = 0$$

◦ Exponential integrators:

$$\begin{aligned} u(t) &= \mathrm{e}^{-t\partial_x^3} u(0) - \frac{1}{2}\mathrm{e}^{-t\partial_x^3}\partial_x \int_0^t \mathrm{e}^{s\partial_x^3} u^2(s)\mathrm{d}s \\ &= \mathrm{e}^{-t\partial_x^3} u(0) - \frac{1}{2}\mathrm{e}^{-t\partial_x^3}\partial_x \int_0^t \mathrm{e}^{s\partial_x^3}\mathrm{d}s \, \left(u^2(0)\right) + \mathcal{O}\left(t^2 \partial_x \partial_t u\right) \end{aligned}$$

Numerical approximation: $u(0) \mapsto u^1$

$$u^1 = \mathrm{e}^{-\tau\partial_x^3} u(0) - \frac{1}{2}\tau \mathrm{e}^{-\tau\partial_x^3}\partial_x \varphi_1(\tau\partial_x^3)\left(u^2(0)\right), \quad \varphi_1(z) = \frac{\mathrm{e}^z - 1}{z}$$

The Korteweg–de Vries equation

$$\partial_t u(t,x) + \partial_x^3 u(t,x) + \frac{1}{2}\partial_x u^2(t,x) = 0$$

◦ Exponential integrators:

$$\begin{aligned} u(t) &= \mathrm{e}^{-t\partial_x^3} u(0) - \frac{1}{2}\mathrm{e}^{-t\partial_x^3}\partial_x \int_0^t \mathrm{e}^{s\partial_x^3} u^2(s)\mathrm{d}s \\ &= \mathrm{e}^{-t\partial_x^3} u(0) - \frac{1}{2}\mathrm{e}^{-t\partial_x^3}\partial_x \int_0^t \mathrm{e}^{s\partial_x^3}\mathrm{d}s\ \left(u^2(0)\right) + \mathcal{O}\left(t^2 \partial_x \partial_t u\right) \end{aligned}$$

Numerical approximation: $u(0) \mapsto u^1$

$$u^1 = \mathrm{e}^{-\tau\partial_x^3} u(0) - \frac{1}{2}\tau \mathrm{e}^{-\tau\partial_x^3}\partial_x \varphi_1(\tau\partial_x^3)\left(u^2(0)\right), \quad \varphi_1(z) = \frac{\mathrm{e}^z - 1}{z}$$

- Requires smooth solutions
- Error analysis difficult (at least for me): discrete Bourgain spaces

The problem with classical methods

Structure of KdV solution: (Duhamel's formula)

$$u(t) = \mathrm{e}^{-t\partial_x^3} u(0) - \frac{1}{2} \mathrm{e}^{-t\partial_x^3} \partial_x \int_0^t \mathrm{e}^{s\partial_x^3} \left(u^2(s) \right) \mathrm{d}s \qquad \text{(KdV flow)}$$

The problem with classical methods

Structure of KdV solution: (Duhamel's formula)

$$u(t) = \mathrm{e}^{-t\partial_x^3}u(0) - \frac{1}{2}\mathrm{e}^{-t\partial_x^3}\partial_x \int_0^t \mathrm{e}^{s\partial_x^3}\left(u^2(s)\right)\mathrm{d}s \qquad \text{(KdV flow)}$$

$$= \mathrm{e}^{-t\partial_x^3}u(0) - \frac{1}{2}\mathrm{e}^{-t\partial_x^3}\partial_x \int_0^t \mathrm{e}^{s\partial_x^3}\left(\mathrm{e}^{-s\partial_x^3}u(0)\right)^2\mathrm{d}s + \int_0^t\int_0^s \dots$$

The problem with classical methods

Structure of KdV solution: (Duhamel's formula)

$$u(t) = \mathrm{e}^{-t\partial_x^3}u(0) - \frac{1}{2}\mathrm{e}^{-t\partial_x^3}\partial_x \int_0^t \mathrm{e}^{s\partial_x^3}\left(u^2(s)\right)\mathrm{d}s \qquad \text{(KdV flow)}$$

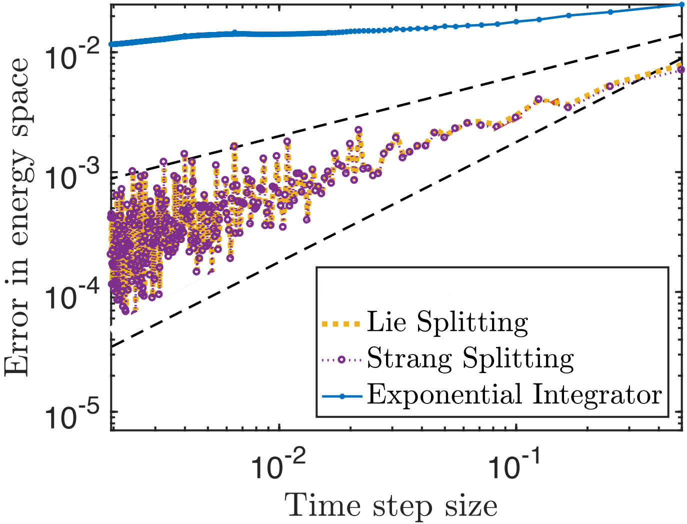
$$= \mathrm{e}^{-t\partial_x^3}u(0) - \frac{1}{2}\mathrm{e}^{-t\partial_x^3}\partial_x \int_0^t \mathrm{e}^{s\partial_x^3}\left(\mathrm{e}^{-s\partial_x^3}u(0)\right)^2\mathrm{d}s + \int_0^t\int_0^s \dots$$

Classical methods linearise frequency interactions (here $f(u) = u^2$)

(splitting) $$\mathrm{e}^{s\partial_x^3} f\left(\mathrm{e}^{-s\partial_x^3}v\right) \approx f(v)$$

(exponential) $$f\left(\mathrm{e}^{-s\partial_x^3}v\right) \approx f(v)$$

Numerical experiment for 'non-smooth' data (Schrödinger with H^1 data)



Classical order (dashed lines) : one and two

Nonlinear idea : KdV solution with periodic b.c.

$$u(t) = \mathrm{e}^{-t\partial_x^3}u(0) - \frac{1}{2}\mathrm{e}^{-t\partial_x^3}\partial_x \int_0^t \mathrm{e}^{s\partial_x^3}\left(\mathrm{e}^{-s\partial_x^3}u(0)\right)^2 \mathrm{d}s + \int_0^t \int_0^s \dots$$

Nonlinear idea : KdV solution with periodic b.c.

$$u(t) = \mathrm{e}^{-t\partial_x^3}u(0) - \frac{1}{2}\mathrm{e}^{-t\partial_x^3}\partial_x \int_0^t \mathrm{e}^{s\partial_x^3}\left(\mathrm{e}^{-s\partial_x^3}u(0)\right)^2 \mathrm{d}s + \int_0^t \int_0^s \dots$$

Nonlinear idea : KdV solution with periodic b.c.

$$u(t) = \mathrm{e}^{-t\partial_x^3} u(0) - \frac{1}{2}\mathrm{e}^{-t\partial_x^3}\partial_x \int_0^t \mathrm{e}^{s\partial_x^3}\left(\mathrm{e}^{-s\partial_x^3}u(0)\right)^2 \mathrm{d}s + \int_0^t \int_0^s \ldots$$

Resonances as a computational tool:

$$\mathrm{Osc}^{\mathrm{KdV}}(v,t) = \partial_x \int_0^t \mathrm{e}^{s\partial_x^3}\left(\mathrm{e}^{-s\partial_x^3}v\right)^2 \mathrm{d}s = \sum_{k+\ell=m} \hat{v}_k \hat{v}_\ell \mathrm{e}^{imx} im \int_0^t \mathrm{e}^{is\mathrm{R}(k,\ell)} \mathrm{d}s$$

$$\mathrm{R}(k,\ell) = -m^3 + k^3 + \ell^3$$

Nonlinear idea : KdV solution with periodic b.c.

$$u(t) = \mathrm{e}^{-t\partial_x^3} u(0) - \frac{1}{2}\mathrm{e}^{-t\partial_x^3}\partial_x \int_0^t \mathrm{e}^{s\partial_x^3}\left(\mathrm{e}^{-s\partial_x^3}u(0)\right)^2 \mathrm{d}s + \int_0^t\int_0^s \dots$$

Resonances as a computational tool:

$$\mathrm{Osc}^{\mathrm{KdV}}(v,t) = \partial_x \int_0^t \mathrm{e}^{s\partial_x^3}\left(\mathrm{e}^{-s\partial_x^3}v\right)^2 \mathrm{d}s = \sum_{k+\ell=m} \hat{v}_k \hat{v}_\ell \mathrm{e}^{imx} im \int_0^t \mathrm{e}^{is\mathrm{R}(k,\ell)}\mathrm{d}s$$

$$\mathrm{R}(k,\ell) = -m^3 + k^3 + \ell^3 = -3k\ell m \quad \text{(factorisation of frequencies)}$$

- integrate $\mathrm{R}(k,\ell)$ exactly + map back to physical space (numerics !)

Nonlinear idea : KdV solution with periodic b.c.

$$u(t) = \mathrm{e}^{-t\partial_x^3} u(0) - \frac{1}{2}\mathrm{e}^{-t\partial_x^3}\partial_x \int_0^t \mathrm{e}^{s\partial_x^3}\left(\mathrm{e}^{-s\partial_x^3}u(0)\right)^2 \mathrm{d}s + \int_0^t \int_0^s \dots$$

Resonances as a computational tool:

$$\mathrm{Osc}^{\mathrm{KdV}}(v,t) = \partial_x \int_0^t \mathrm{e}^{s\partial_x^3}\left(\mathrm{e}^{-s\partial_x^3}v\right)^2 \mathrm{d}s = \sum_{k+\ell=m} \hat{v}_k \hat{v}_\ell \mathrm{e}^{imx} im \int_0^t \mathrm{e}^{is\mathrm{R}(k,\ell)}\mathrm{d}s$$

$$\mathrm{R}(k,\ell) = -m^3 + k^3 + \ell^3 = -3k\ell m \quad \text{(factorisation of frequencies)}$$

- integrate $\mathrm{R}(k,\ell)$ exactly + map back to physical space (numerics !)

Scheme:

$$u^n \mapsto u^{n+1} = \mathrm{e}^{-\tau\partial_x^3} u^n - \frac{1}{6}\left[\partial_x^{-1}\left(\mathrm{e}^{-\tau\partial_x^3}u^n\right)^2 - \mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}(u^n)^2\right]$$

Nonlinear idea : KdV solution with periodic b.c.

$$u(t) = \mathrm{e}^{-t\partial_x^3} u(0) - \frac{1}{2}\mathrm{e}^{-t\partial_x^3}\partial_x \int_0^t \mathrm{e}^{s\partial_x^3}\left(\mathrm{e}^{-s\partial_x^3}u(0)\right)^2 \mathrm{d}s + \int_0^t\int_0^s \ldots$$

Resonances as a computational tool:

$$\mathsf{Osc}^{\mathsf{KdV}}(v,t) = \partial_x \int_0^t \mathrm{e}^{s\partial_x^3}\left(\mathrm{e}^{-s\partial_x^3}v\right)^2 \mathrm{d}s = \sum_{k+\ell=m} \hat{v}_k \hat{v}_\ell \mathrm{e}^{imx} im \int_0^t \mathrm{e}^{is\mathsf{R}(k,\ell)}\mathrm{d}s$$

$\mathsf{R}(k,\ell) = -m^3 + k^3 + \ell^3 = -3k\ell m$ (factorisation of frequencies)

- integrate $\mathsf{R}(k,\ell)$ exactly + map back to physical space (numerics !)

Comparison with classical methods:

Splitting method: $\mathsf{R}(k,\ell) \approx 0$ for all $k,\ell \in \mathbb{Z}$

Exponential integrator: $\mathsf{R}(k,\ell) \approx (k+\ell)^3$ for all $k,\ell \in \mathbb{Z}$

Nonlinear idea (for KdV)

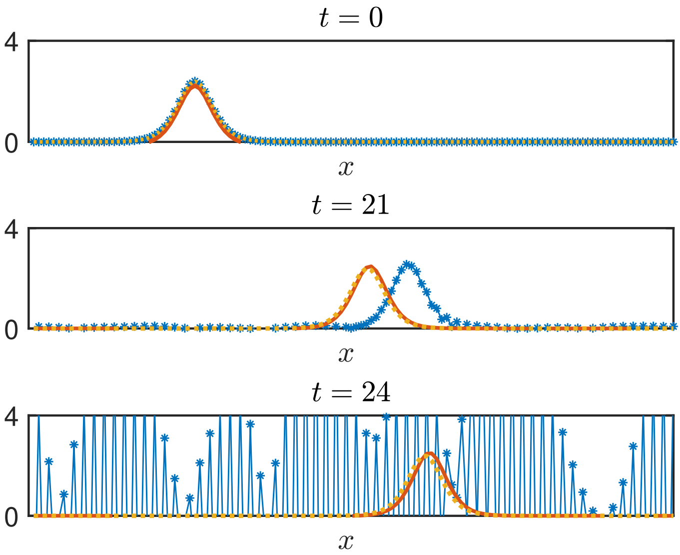


Fig.: simulation of solitary wave solution (orange):
$*$ classical method, -- resonance based approach

What about a general class of (dispersive) eqs?

Dispersive PDE $\quad \partial_t u - \mathrm{i}\mathcal{L}(\nabla, \frac{1}{\varepsilon})u = f(u) \quad + \quad$ i.c. and b.c.

Classical methods: linearise frequency interactions, e.g.,

$$\text{(splitting)} \quad \mathrm{e}^{-\mathrm{i}\xi\mathcal{L}} f\left(\mathrm{e}^{\mathrm{i}\xi\mathcal{L}} v\right) \approx f(v)$$

$$\text{(exponential)} \quad f\left(\mathrm{e}^{\mathrm{i}\xi\mathcal{L}} v\right) \approx f(v)$$

What about a general class of (dispersive) eqs?

Dispersive PDE $\quad \partial_t u - \mathrm{i}\mathcal{L}(\nabla, \frac{1}{\varepsilon})u = f(u) \quad + \;$ i.c. and b.c.

Classical methods: linearise frequency interactions, e.g.,

(splitting) $\quad \mathrm{e}^{-\mathrm{i}\xi\mathcal{L}} f\left(\mathrm{e}^{\mathrm{i}\xi\mathcal{L}} v\right) \approx f(v)$

(exponential) $\quad f\left(\mathrm{e}^{\mathrm{i}\xi\mathcal{L}} v\right) \approx f(v)$

Nonlinear improvement:

$$\mathrm{e}^{-\mathrm{i}\xi\mathcal{L}} f\left(\mathrm{e}^{\mathrm{i}\xi\mathcal{L}} v\right) = \left[\mathrm{e}^{i\xi\mathcal{L}(\nabla, \frac{1}{\varepsilon})_{\text{dominant}}} f_{\text{dom}}(v)\right] f_{\text{non-oscillatory}}(v) + \text{l.o.t.}$$

Key: Choice of $\mathcal{L}_{\text{dominant}}$?

Resonances as a computational tool

Equation $(\mathcal{L}, f)$ (classical numerics)		Resonances[a] $R(k)$ (LAHACODE method)
NLS	$\mathrm{i}\partial_t u + \Delta u = \|u\|^2 u$	$-2k^2 + 2k(\ell + m) + 2\ell m$
KdV	$\partial_t u + \partial_x^3 u = \partial_x(u^2)$	$3k\ell(k+\ell)$
KG	$\varepsilon^2 \partial_t^2 u - \Delta u + \varepsilon^{-2} u = \|u\|^2 u$	$\sum_{\lambda=\pm 2,4} e^{\mathrm{i}t\lambda\frac{1}{\varepsilon^2}} u^{\lambda}_{\text{non-oscillatory}}$

What about a general class of (dispersive) eqs?

Dispersive PDE $\partial_t u - \mathrm{i}\mathcal{L}(\nabla, \frac{1}{\varepsilon})u = f(u)$ + i.c. and b.c.

Nonlinear improvement:

$$\underbrace{\mathrm{e}^{-\mathrm{i}\xi\mathcal{L}} f\left(\mathrm{e}^{\mathrm{i}\xi\mathcal{L}}\right.}\left. v\right) = \left[\mathrm{e}^{i\xi\mathcal{L}(\nabla,\frac{1}{\varepsilon})_{\text{dominant}}} f_{\text{dom}}(v)\right] f_{\text{non-oscillatory}}(v) + \text{l.o.t.}$$

Key: Choice of $\mathcal{L}_{\text{dominant}}$?

Resonances as a computational tool

What about a general class of (dispersive) eqs?

Dispersive PDE $\partial_t u - \mathrm{i}\mathcal{L}(\nabla, \frac{1}{\varepsilon})u = f(u)$ + i.c. and b.c.

Nonlinear improvement:

$$\mathrm{e}^{-\mathrm{i}\xi\mathcal{L}} f\left(\mathrm{e}^{\mathrm{i}\xi\mathcal{L}} v\right) = \left[\mathrm{e}^{i\xi\mathcal{L}(\nabla,\frac{1}{\varepsilon})_{\text{dominant}}} f_{\text{dom}}(v)\right] f_{\text{non-oscillatory}}(v) + \text{l.o.t.}$$

Key: Choice of $\mathcal{L}_{\text{dominant}}$?

Resonances as a computational tool

Example: cubic NLS $i\partial_t u = -\Delta u + |u|^2 u$ with $x \in \mathbb{T}^d$

$$\int_0^t e^{-i\xi\Delta} f\left(e^{i\xi\Delta} v\right) \mathrm{d}\xi = \sum_{k=k_1-k_2+k_3\in\mathbb{Z}^d} \hat{v}_{k_1} \overline{\hat{v}}_{k_2} \hat{v}_{k_3} e^{ikx} \cdot \int_0^t e^{i(k^2-k_1^2+k_2^2-k_3^2)\xi} \mathrm{d}\xi$$

What about a general class of (dispersive) eqs?

Dispersive PDE $\quad \partial_t u - \mathrm{i}\mathcal{L}(\nabla, \frac{1}{\varepsilon})u = f\big(u\big) \quad + \quad$ i.c. and b.c.

Nonlinear improvement:

$$\mathrm{e}^{-\mathrm{i}\xi\mathcal{L}} f\left(\mathrm{e}^{\mathrm{i}\xi\mathcal{L}} v\right) = \left[\mathrm{e}^{i\xi\mathcal{L}(\nabla,\frac{1}{\varepsilon})_{\text{dominant}}} f_{\text{dom}}(v)\right] f_{\text{non-oscillatory}}(v) + \text{l.o.t.}$$

Key: Choice of $\mathcal{L}_{\text{dominant}}$?

Resonances as a computational tool

Example: cubic NLS $i\partial_t u = -\Delta u + |u|^2 u$ with $x \in \mathbb{T}^d$

$$\int_0^t e^{-i\xi\Delta} f\left(e^{i\xi\Delta} v\right) \mathrm{d}\xi = \sum_{k=k_1-k_2+k_3 \in \mathbb{Z}^d} \hat{v}_{k_1} \overline{\hat{v}}_{k_2} \hat{v}_{k_3} e^{ikx} \cdot \int_0^t e^{i(k^2-k_1^2+k_2^2-k_3^2)\xi} \mathrm{d}\xi$$

Q: Dominant part in $\mathrm{R}(k_1, k_2, k_3) = 2k_2^2 + 2k_1k_3 - 2k_2(k_1 + k_3)$?

What about a general class of (dispersive) eqs?

Dispersive PDE $\partial_t u - i\mathcal{L}(\nabla, \frac{1}{\varepsilon})u = f(u)$ + i.c. and b.c.

Nonlinear improvement:

$$e^{-i\xi\mathcal{L}} f\left(e^{i\xi\mathcal{L}} v\right) = \left[e^{i\xi\mathcal{L}(\nabla,\frac{1}{\varepsilon})_{\text{dominant}}} f_{\text{dom}}(v)\right] f_{\text{non-oscillatory}}(v) + \text{l.o.t.}$$

Key: Choice of $\mathcal{L}_{\text{dominant}}$?

Resonances as a computational tool

Example: cubic NLS $i\partial_t u = -\Delta u + |u|^2 u$ with $x \in \mathbb{T}^d$

$$\int_0^t e^{-i\xi\Delta} f\left(e^{i\xi\Delta} v\right) \mathrm{d}\xi = \sum_{k=k_1-k_2+k_3 \in \mathbb{Z}^d} \hat{v}_{k_1} \overline{\hat{v}}_{k_2} \hat{v}_{k_3} e^{ikx} \cdot \int_0^t e^{i(k^2-k_1^2+k_2^2-k_3^2)\xi} \mathrm{d}\xi$$

Q: Dominant part in $\mathrm{R}(k_1, k_2, k_3) = 2k_2^2 + 2k_1k_3 - 2k_2(k_1+k_3)$

Numerical example: Talbot effect

Cubic Schrödinger equation with step function initial value

For times $t = \pi\mathbb{Q}$: We observe quantisation

Analysis for linear dispersive eqs: K. Oskolkov, P. Olver, ...
for 1d, periodic cubic Schrödinger eq: M.B. Erdogan, N. Tzirakis

A lot of Questions

Bottleneck of resonances as a computational tool

Equation $(\mathcal{L}, f)$ (classical numerics)	Resonances[a] $\mathrm{R}(k)$ (LAHACODE method)
NLS $i\partial_t u + \Delta u = \lvert u\rvert^2 u$	$-2k^2 + 2k(\ell + m) + 2\ell m$
KdV $\partial_t u + \partial_x^3 u = \partial_x(u^2)$	$3k\ell(k+\ell)$
KG $\varepsilon^2 \partial_t^2 u - \Delta u + \varepsilon^{-2} u = \lvert u\rvert^2 u$	$\sum_{\lambda=\pm 2,4} e^{it\lambda\frac{1}{\varepsilon^2}} u^{\lambda}_{\text{non-oscillatory}}$

Q1 : Can we find an overarching algorithm ?

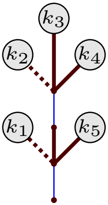
Q2 : Error estimates at low regularity ?

Q3 : More general class of equations ?

Q4 : Structure preservation, long time scales, . . . ?

Part 1 : Overarching algorithm

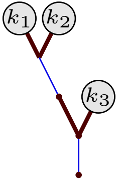
[Bruned–S, Forum of Mathematics Pi '22]



Framework for periodic dispersive pdes

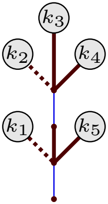
$$\partial_t u + i\mathcal{L}\left(\nabla, \frac{1}{\varepsilon}\right) = f(u)$$

via decorated trees



Part 1 : Overarching algorithm

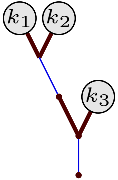
[Bruned–S, Forum of Mathematics Pi '22]



Framework for periodic dispersive pdes

$$\partial_t u + i\mathcal{L}\left(\nabla, \frac{1}{\varepsilon}\right) = f(u)$$

via decorated trees

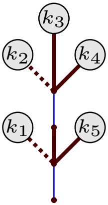


Difficulty: control of frequency interactions

$$u(t) = e^{it\mathcal{L}}u_0 + e^{it\mathcal{L}} \int_0^t e^{-i\xi_1\mathcal{L}} f(e^{i\xi_1\mathcal{L}}u_0) d\xi_1$$

Part 1 : Overarching algorithm

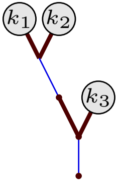
[Bruned–S, Forum of Mathematics Pi '22]



Framework for periodic dispersive pdes

$$\partial_t u + i\mathcal{L}\left(\nabla, \frac{1}{\varepsilon}\right) = f(u)$$

via decorated trees



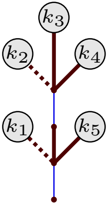
Difficulty: control of frequency interactions

$$u(t) = e^{it\mathcal{L}}u_0 + e^{it\mathcal{L}}\int_0^t e^{-i\xi_1\mathcal{L}} f(e^{i\xi_1\mathcal{L}}u_0)d\xi_1$$

$$+ e^{it\mathcal{L}}\int_0^t e^{-i\xi_1\mathcal{L}}\left[f'(e^{i\xi_1\mathcal{L}}u_0)e^{i\xi_1\mathcal{L}}\int_0^{\xi_1} e^{-i\xi_2\mathcal{L}} f(e^{i\xi_2\mathcal{L}}u_0)d\xi_2\right]d\xi_1 + \dots$$

Part 1 : Overarching algorithm

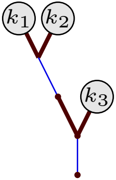
[Bruned–S, Forum of Mathematics Pi '22]



Framework for periodic dispersive pdes

$$\partial_t u + i\mathcal{L}\left(\nabla, \frac{1}{\varepsilon}\right) = f(u)$$

via decorated trees



Difficulty: control of frequency interactions

$$u(t) = e^{it\mathcal{L}}u_0 + e^{it\mathcal{L}}\int_0^t e^{-i\xi_1\mathcal{L}} f(e^{i\xi_1\mathcal{L}}u_0)d\xi_1$$

$$+ e^{it\mathcal{L}}\int_0^t e^{-i\xi_1\mathcal{L}}\left[f'(e^{i\xi_1\mathcal{L}}u_0)e^{i\xi_1\mathcal{L}}\int_0^{\xi_1} e^{-i\xi_2\mathcal{L}} f(e^{i\xi_2\mathcal{L}}u_0)d\xi_2\right]d\xi_1 + \ldots$$

Idea (tree series): $\hat{u}_k(t) = \sum_{T\in\mathcal{V}_k^p} \frac{\Upsilon^f(T)}{S(T)}(\mathcal{I}_p T)(t, u_0) + \mathcal{O}(t^{p+1})$

($\mathcal{I}_p = \mathcal{I}_{t,\xi_1,..,\xi_p}$ integral operator, T trees with leaf decoration k_ℓ)

Part 1 : Overarching algorithm

[Bruned–S, Forum of Mathematics Pi '22]

Difficulty: control of frequency interactions

$$u(t) = e^{it\mathcal{L}}u_0 + e^{it\mathcal{L}}\int_0^t e^{-i\xi_1\mathcal{L}} f(e^{i\xi_1\mathcal{L}}u_0)d\xi_1$$

$$+ e^{it\mathcal{L}}\int_0^t e^{-i\xi_1\mathcal{L}}\left[f'(e^{i\xi_1\mathcal{L}}u_0)e^{i\xi_1\mathcal{L}}\int_0^{\xi_1} e^{-i\xi_2\mathcal{L}} f(e^{i\xi_2\mathcal{L}}u_0)d\xi_2\right]d\xi_1 + \dots$$

Idea (tree series): $\hat{u}_k(t) = \sum_{T\in\mathcal{V}_k^p} \frac{\Upsilon^f(T)}{S(T)}(\mathcal{I}_p T)(t, u(0)) + \mathcal{O}(t^{p+1})$

Part 1 : Overarching algorithm

[Bruned–S, Forum of Mathematics Pi '22]

Difficulty: control of frequency interactions

$$u(t) = e^{it\mathcal{L}}u_0 + e^{it\mathcal{L}}\int_0^t e^{-i\xi_1\mathcal{L}} f(e^{i\xi_1\mathcal{L}}u_0)d\xi_1$$
$$+ e^{it\mathcal{L}}\int_0^t e^{-i\xi_1\mathcal{L}}\left[f'(e^{i\xi_1\mathcal{L}}u_0)e^{i\xi_1\mathcal{L}}\int_0^{\xi_1} e^{-i\xi_2\mathcal{L}} f(e^{i\xi_2\mathcal{L}}u_0)d\xi_2\right]d\xi_1 + \dots$$

Idea (tree series): $\hat{u}_k(t) = \sum_{T\in\mathcal{V}_k^p} \frac{\Upsilon^f(T)}{S(T)}(\mathcal{I}_p T)(t, u(0)) + \mathcal{O}(t^{p+1})$

Discretisation operator $\mathcal{I}_p \approx \mathcal{I}_p^{\mathsf{d}}$ (in spirit of resonance approach)

$$\mathcal{I}_p^{(\mathsf{d})} = \left(\hat{\mathcal{I}}_p^{(\mathsf{d})} \otimes A_p^{(\mathsf{d})}\right)\Delta \qquad \text{(Birkhoff-type factorisation)}$$

structure close to SPDEs with Regularity Structures [M. Hairer 2014]

Part 2 : Sharp error estimates in low regularity spaces

[Ostermann–Rousset–S, JEMS, FoCM '22]

Continuous level: Strichartz estimates (Ginibre–Velo, Keel–Tao)

$$\|e^{\mathrm{i}t\Delta}v\|_{L_t^p L_x^q(\mathbb{R}^d)} \leq c_{d,q,p}\|v\|_{L_x^2(\mathbb{R}^d)} \quad 2\leq p,q\leq\infty,\ \frac{2}{p}+\frac{d}{q}=\frac{d}{2},\ (p,q,d)\neq(2,\infty,2)$$

Part 2 : Sharp error estimates in low regularity spaces

[Ostermann–Rousset–S, JEMS, FoCM '22]

Continuous level: Strichartz estimates (Ginibre–Velo, Keel–Tao)

$$\|e^{it\Delta} v\|_{L^p_t L^q_x(\mathbb{R}^d)} \leq c_{d,q,p} \|v\|_{L^2_x(\mathbb{R}^d)} \quad {\scriptstyle 2\leq p,q\leq\infty,\ \frac{2}{p}+\frac{d}{q}=\frac{d}{2},\ (p,q,d)\neq(2,\infty,2)}$$

Finite dimensional (discrete) counterpart ?

Part 2 : Sharp error estimates in low regularity spaces

[Ostermann–Rousset–S, JEMS, FoCM '22]

Continuous level: Strichartz estimates (Ginibre–Velo, Keel–Tao)

$$\|e^{it\Delta}v\|_{L^p_t L^q_x(\mathbb{R}^d)} \leq c_{d,q,p}\|v\|_{L^2_x(\mathbb{R}^d)} \quad 2\leq p,q\leq\infty,\ \tfrac{2}{p}+\tfrac{d}{q}=\tfrac{d}{2},\ (p,q,d)\neq(2,\infty,2)$$

Finite dimensional (discrete) counterpart ?

Ignat–Zuazua: L^2 error estimates for cubic NLS: $u \in H^2(\mathbb{R}^d)$, $d \leq 3$

Theorem (Ostermann–Rousset–S): For every (p,q) admissible $(p > 2)$ $\exists$ $C > 0$ s.t. for $K = \tau^{-\alpha/2}$ with $\|F\|_{l^p_\tau L^q} = \left(\tau \sum_{k\in\mathbb{Z}} \|F_k\|^p_{L^q}\right)^{1/p}$ and $\alpha \geq 1$

$$\left\|e^{it_n\Delta}\Pi_K v\right\|_{l^p_\tau L^q_x(\mathbb{R}^d)} \leq c_{d,q,p}\tau^{p(1-\alpha)}\|v\|_{L^2_x(\mathbb{R}^d)} \qquad (\text{discrete } t_n = n\tau)$$

L^2 error estimates for $u \in H^\sigma, \sigma > 0$ $\qquad (u \xrightarrow{p} u_{\Pi_K} \xrightarrow{t_n} u^n_{\Pi_K})$

$$\|u(t_n) - u^n_{\Pi_K}\|_{L^2(\mathbb{R}^d)} \leq \tau^\delta c(T, \|u\|_{H^\sigma(\mathbb{R}^d)}), \quad \delta = \delta(d,\sigma,\alpha)$$

($\mathbb{T}$, $\mathbb{T}^2$: discrete Bourgain $\|\Pi_K u_n\|_{l^4_\tau L^4(\mathbb{T})} \lesssim \left(K\tau^{1/2}\right)^{1/2} \|u_n\|_{X^{0,\frac{3}{8}}_\tau}$)

Part 3 : Non periodic b.c., general class of equations

[Rousset–S, Li–Ma–S, SIAM J. Numer. Anal. '21, '22]

Model problem:

$$\partial_t u + \mathcal{L}u = f(u,\overline{u}) \qquad (t,x) \in \mathbb{R} \times \Omega \subset \mathbb{R}^d$$

- $\mathcal{L}$ generates contractive $\mathcal{C}_0$ semigroup on X
- $-\mathcal{L} + \overline{\mathcal{L}}$ generates unitary group on X
- $f(u,\overline{u}) = \mathcal{B}\left(F(u) \cdot G(\overline{u})\right)$, F, G : $\mathbb{C} \to \mathbb{C}^J$ smooth, $\mathcal{B}$ linear

Ex.: NLS, Ginzburg-Landau, (half-)wave, Navier–Stokes, etc.

Part 3 : Non periodic b.c., general class of equations

[Rousset–S, Li–Ma–S, SIAM J. Numer. Anal. '21, '22]

Model problem:

$$\partial_t u + \mathcal{L} u = f(u, \overline{u}) \qquad (t, x) \in \mathbb{R} \times \Omega \subset \mathbb{R}^d$$

- $\mathcal{L}$ generates contractive $\mathcal{C}_0$ semigroup on X
- $-\mathcal{L} + \overline{\mathcal{L}}$ generates unitary group on X
- $f(u, \overline{u}) = \mathcal{B}\left(F(u) \cdot G(\overline{u})\right)$, F, $G : \mathbb{C} \to \mathbb{C}^J$ smooth, $\mathcal{B}$ linear

Ex.: NLS, Ginzburg-Landau, (half-)wave, Navier–Stokes, etc.

Resonance based approach:

$$u^{\ell+1} = e^{\tau \mathcal{L}} \Big(u^\ell + \tau \mathcal{B} \Big(F(u^\ell) \cdot \varphi_1\big(\tau(-\mathcal{L} + \overline{\mathcal{L}})\big) G(\overline{u^\ell}) \Big) \Big)$$

Part 3 : Non periodic b.c., general class of equations

[Rousset–S, Li–Ma–S, SIAM J. Numer. Anal. '21, '22]

Model problem:

$$\partial_t u + \mathcal{L}u = f(u,\overline{u}) \qquad (t,x) \in \mathbb{R} \times \Omega \subset \mathbb{R}^d$$

- $\mathcal{L}$ generates contractive $\mathcal{C}_0$ semigroup on X
- $-\mathcal{L} + \overline{\mathcal{L}}$ generates unitary group on X
- $f(u,\overline{u}) = \mathcal{B}\left(F(u)\cdot G(\overline{u})\right)$, F, G : $\mathbb{C} \to \mathbb{C}^J$ smooth, $\mathcal{B}$ linear

Ex.: NLS, Ginzburg-Landau, (half-)wave, Navier–Stokes, etc.

Resonance based approach:

$$u^{\ell+1} = e^{\tau\mathcal{L}}\Big(u^\ell + \tau\mathcal{B}\Big(F(u^\ell)\cdot\varphi_1\big(\tau(-\mathcal{L}+\overline{\mathcal{L}})\big)G(\overline{u^\ell})\Big)\Big)$$

Convergence* : $\|u(t_n) - u^n\|_X \le c\tau$ if $\|\mathcal{C}[f,\mathcal{L}](u,\overline{u})\|_{L^\infty_{t\le t_n}X} < \infty$

$$\mathcal{C}[f,\mathcal{L}](v,w) = -\mathcal{L}f(v,w) + D_1 f(v,w)\cdot\mathcal{L}v + D_2 f(v,w)\cdot\mathcal{L}w$$

*improves classical convergence results (if $\mathcal{L}$ satisfies Leibniz rule)

Part 4: What about structure preservation, long time scales, etc. ?

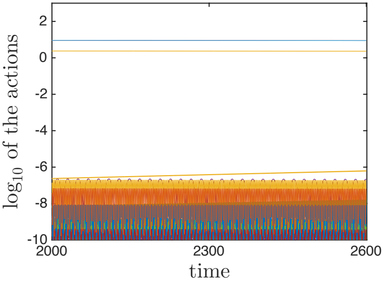
Problem: numerical resonances (= purely discrete artifact) trigger non-physical energy shift from low to high modes (even for C^∞ data)

Part 4: What about structure preservation, long time scales, etc. ?

Problem: numerical resonances (= purely discrete artifact) trigger non-physical energy shift from low to high modes (even for C^∞ data)

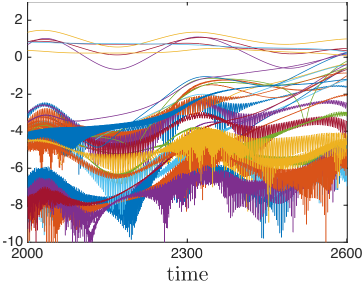
Example: cubic NLS on $\mathbb{T}$ $\quad i\partial_t u = -\partial_x^2 u + |u|^2 u$ with small C^∞ data

Splitting method:



non-resonant

$(\Delta t = 0.315)$



resonant

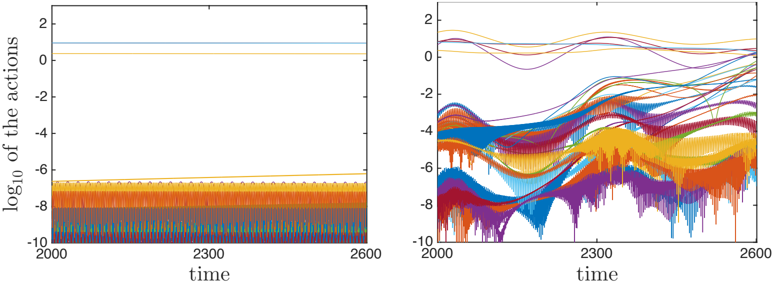
$(\Delta t = 0.1 \times \pi \approx 0.314159)$

Part 4: What about structure preservation, long time scales, etc. ?

Problem: numerical resonances (= purely discrete artifact) trigger non-physical energy shift from low to high modes (even for C^∞ data)

Example: cubic NLS on $\mathbb{T}$ $\quad i\partial_t u = -\partial_x^2 u + |u|^2 u$ with small C^∞ data

Splitting method:

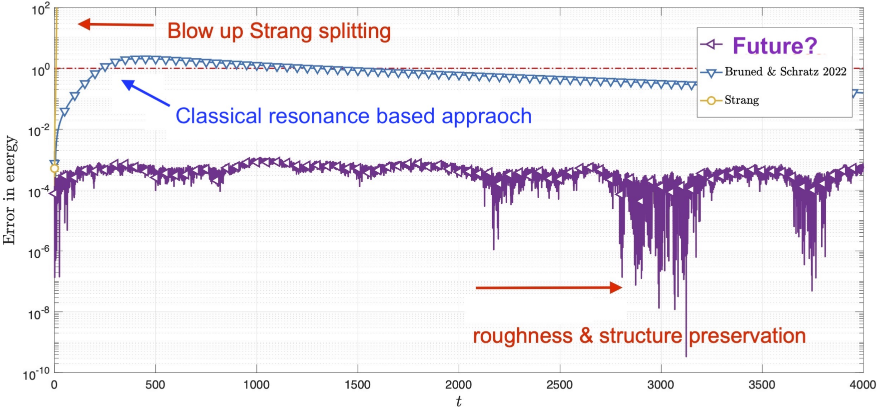


non-resonant $(\Delta t = 0.315)$ > resonant $(\Delta t = 0.1 \times \pi \approx 0.314159)$

Part 4: What about structure preservation, long time scales, etc. ?

...gets even worse for rough data !

Example: cubic NLS on $\mathbb{T}$ with $\mathcal{O}(1)$ rough data



[Alama Bronsard–Bruned–Maierhofer–Schratz, arXiv:2305.16737]

Still a lot of open Questions

Q1 : Structure preservation for rough data?

Q2 : Non-smooth phenomena on long time scales
(e.g., blow up, growth of Sobolev norms, wave turbulence, . . .) ?

How far can we actually go at the discrete level :
down with regularity and up in time scales ?