The Continuum Hypothesis

W. Hugh Woodin

Harvard University

June 20, 2019

Ordinals: the transfinite numbers

- $\emptyset$ is the smallest ordinal: this is 0.
- $\{\emptyset\}$ is the next ordinal: this is 1.
- $\{\emptyset, \{\emptyset\}\}$ is next ordinal: this is 2.

If α is an ordinal then

- α is just the set of all ordinals β such that β is smaller than α,
- $\alpha + 1 = \alpha \cup \{\alpha\}$ is the next largest ordinal.

ω denotes the least infinite ordinal, it is the set of all finite ordinals.

V: The Universe of Sets

The power set

Suppose X is a set. The **powerset** of X is the set

$$\mathcal{P}(X) = \{Y \mid Y \text{ is a subset of } X\}.$$

Cumulative Hierarchy of Sets

The universe V of sets is generated by defining V_α by induction on the ordinal α:

1. $V_0 = \emptyset$,
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$,
3. if α is a limit ordinal then $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$.

- If X is a set then $X \in V_\alpha$ for some ordinal α.

- $V_0 = \emptyset$, $V_1 = \{\emptyset\}$, $V_2 = \{\emptyset, \{\emptyset\}\}$.
  - These are just the ordinals: 0, 1, and 2.
- V_3 has 4 elements (and is clearly not an ordinal).
- V_4 has 16 elements.
- V_5 has $65,536$ elements.
- V_{1000} has a lot of elements.

V_ω is infinite, it is the set of all (hereditarily) finite sets.

The structure $(V_\omega, \in)$ is **mathematically identical** to the structure $(\mathbb{N}, +, \cdot)$:

- Each structure can be interpreted in the other structure.

Cardinals

Definition: cardinals

An ordinal α is a **cardinal** if

$$|\beta| < |\alpha|$$

for all $\beta < \alpha$

- The finite ordinals are cardinals.
- ω is a cardinal.
- For each ordinal α, ω_α denotes the α-th **infinite** cardinal.
  - $\omega_0 = \omega$ and ω_1 is the least uncountable cardinal.
  - $\omega_{\alpha+1}$ is the least cardinal bigger than ω_α.

By the basic axioms of Set Theory including the Axiom of Choice:

- For each ordinal α, ω_α exists.
- For every infinite set X, there is an ordinal α such that

$$|X| = \omega_\alpha.$$

The Continuum Hypothesis: CH

Theorem (Cantor)

The set $\mathbb{N}$ of all natural numbers and the set $\mathbb{R}$ of all real numbers do not have the same cardinality.

Assuming the Axiom of Choice: $|\mathbb{R}| = \omega_\alpha$ for some $\alpha \geq 1$.

- Which α?

Cantor's Continuum Hypothesis

Suppose $A \subseteq \mathbb{R}$ is infinite. Then either:

1. A and $\mathbb{N}$ have the same cardinality, or
2. A and $\mathbb{R}$ have the same cardinality.

- Assuming the Axiom of Choice, the Continuum Hypothesis is equivalent to the assertion: $|\mathbb{R}| = \omega_1$.

Hilbert's First Problem

- The problem of the Continuum Hypothesis is the first problem Hilbert's list of 23 problems from his ICM address of 1900.

- Many tried to solve the problem of the Continuum Hypothesis and failed.

In 1940, Gödel showed that it is consistent with the axioms of Set Theory that the Continuum Hypothesis be true.

- One **cannot refute** the Continuum Hypothesis.

In 1963, on July 4th, Cohen announced in a lecture at Berkeley that it is consistent with the axioms of Set Theory that the Continuum Hypothesis be false.

- One **cannot prove** the Continuum Hypothesis.

Cohen's method

- The ZFC axioms of Set Theory **formally** specify the founding principles for the conception of V.

If M is a model of ZFC then M contains "blueprints" for virtual models N of ZFC, which enlarge M. These blueprints can be constructed and analyzed from within M.

- If M is countable then every blueprint constructed within M can be realized as genuine enlargement of M.

- Cohen proved that **every** model of ZFC contains a blueprint for an enlargement in which the Continuum Hypothesis is false.
- Cohen's method also shows that **every** model of ZFC contains a blueprint for an enlargement in which the Continuum Hypothesis is true.

The extent of Cohen's method: It is not just about CH

A challenging time for the conception of V

- Cohen's method has been vastly developed in the 5 decades since Cohen's original work.
- Many problems have been showed to be unsolvable including problems outside Set Theory:
  - **Whitehead Problem** (Group Theory)
    - (Shelah:1974)
  - **Kaplansky's Conjecture** (Analysis)
    - (Dales-Esterle, Solovay:1976)
  - **Suslin's Problem** (Combinatorics of the real line)
    - (Solovay-Tennenbaum, Jensen, Jech:1968)
  - **Borel Conjecture** (Measure Theory)
    - (Laver:1976)
  - **Brown-Douglas-Filmore Automorphism Problem** (Operator Algebras)
    - (Phillips-Weaver, Farah:2011)

Ok, maybe it is just time to give up

Claim

- *Large cardinal axioms are not provable;*
  - by Gödel's Second Incompleteness Theorem.
- **But, large cardinal axioms are falsifiable**.

Prediction

No contradiction from the existence of infinitely many Woodin cardinals will be discovered within the next 1000 *years.*

- **Not by any means whatsoever**.

Truth beyond our formal reach

The real claim of course is:

- There is **no** contradiction from the existence of infinitely many Woodin cardinals.

Claim

- Such statements cannot be **formally** proved.
- This suggests there is a component in the evolution of our understanding of Mathematics which is **not** formal.
  - There is mathematical knowledge which is not entirely based in proofs.

The simplest uncountable sets

Definition

A set $A \subseteq V_{\omega+1}$ is a **projective set** if:

- A can be logically defined in the structure
$$(V_{\omega+1}, \in)$$
from parameters.

Definition

A set $A \subseteq V_{\omega+1} \times V_{\omega+1}$ is a **projective set** if:

- A can be logically defined as a binary relation in the structure
$$(V_{\omega+1}, \in)$$
from parameters.

The Continuum Hypothesis and the Projective Sets

The Continuum Hypothesis

Suppose $A \subseteq V_{\omega+1}$ is infinite. Then either:

1. A and V_ω have the same cardinality, or
2. A and $V_{\omega+1}$ have the same cardinality.

The projective Continuum Hypothesis

Suppose $A \subseteq V_{\omega+1}$ is an infinite projective set. Then either:

1. A and V_ω have the same cardinality, or
2. There is a bijection
$$F : V_{\omega+1} \to A$$
such that F is a projective set.

The Axiom of Choice

Definition

Suppose that

$$A \subseteq X \times Y$$

A function

$$F : X \to Y$$

is a **choice function** for A if for all $a \in X$:

- If there exists $b \in Y$ such that $(a, b) \in A$ then $(a, F(a)) \in A$.

The Axiom of Choice

For every set

$$A \subseteq X \times Y$$

there exists a choice function for A.

The Axiom of Choice and the Projective Sets

The projective Axiom of Choice

Suppose $A \subseteq V_{\omega+1} \times V_{\omega+1}$ is a projective set. Then there is a function

$$F : V_{\omega+1} \to V_{\omega+1}$$

such that:

- F is a choice function for A.
- F is a projective set.

These are both also unsolvable problems

The actual constructions of Gödel and Cohen show that both problems are formally unsolvable.

- In Gödel's universe L:
  - The projective Axiom of Choice holds.
  - The projective Continuum Hypothesis holds.
- In the Cohen enlargement of L (as given by the actual blueprint which Cohen defined for the failure of CH):
  - The projective Axiom of Choice is false.
  - The projective Continuum Hypothesis is false.

Beyond the basic axioms: large cardinal axioms

Sharpening the conception of V

- The ZFC axioms are naturally augmented by additional axioms which assert the existence of “very large” infinite sets.
  - Such axioms assert the existence of **large cardinals**.

These large cardinals include:

- Measurable cardinals
- Woodin cardinals
- Superstrong cardinals
- Supercompact cardinals
- Extendible cardinals
- Huge cardinals
- Axiom I_0 cardinals.

An unexpected entanglement

Theorem (1984)

Suppose there are infinitely many Woodin cardinals. Then:

- *The projective Continuum Hypothesis holds.*

Theorem (1985: Martin-Steel)

Suppose there are infinitely many Woodin cardinals. Then:

- *The projective Axiom of Choice holds.*

We now have the correct conception of $V_{\omega+1}$ and the projective sets.

- This conception yields axioms for the projective sets.
- These (determinacy) axioms in turn are closely related to (and follow from) large cardinal axioms.

Logical definability

The definable power set

For each set X, $\mathcal{P}_{\mathrm{Def}}(X)$ denotes the set of all $Y \subseteq X$ such that Y is logically definable in the structure $(X, \in)$ from parameters in X.

The collection of all the projective subsets of $V_{\omega+1}$ is exactly given by:

$$\mathcal{P}_{\mathrm{Def}}(V_{\omega+1})$$

The effective cumulative hierarchy: L

Cumulative Hierarchy of Sets

The cumulative hierarchy is defined by induction on α as follows.

1. $V_0 = \emptyset$.
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$.
3. if α is a limit ordinal then $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$.

- V is the class of all sets X such that $X \in V_\alpha$ for some α.

Gödel's constructible universe, L

Define L_α by induction on α as follows.

1. $L_0 = \emptyset$.
2. $L_{\alpha+1} = \mathcal{P}_{\mathrm{Def}}(L_\alpha)$.
3. if α is a limit ordinal then $L_\alpha = \cup\{L_\beta \mid \beta < \alpha\}$.

- L is the class of all sets X such that $X \in L_\alpha$ for some α.

The missing axiom for V?

The axiom: $V = L$

Suppose X is a set. Then $X \in L$.

Theorem (Gödel:1940)

Assume $V = L$. Then the Continuum Hypothesis holds.

- Suppose there is a Cohen-blueprint for $V = L$. Then:
  - the axiom $V = L$ must hold and the blueprint is trivial.

Claim

Adopting the axiom $V = L$ completely negates the ramifications of Cohen's method.

The axiom $V = L$ and large cardinals

Theorem (Scott:1961)

Assume $V = L$. Then there are no measurable cardinals.

- *In fact there are no (genuine) large cardinals.*

- Assume $V = L$. **Then there are no Woodin cardinals.**

Clearly:

The axiom $V = L$ is false.

Universally Baire sets

Definition (Feng-Magidor-Woodin)

A set $A \subseteq \mathbb{R}^n$ is **universally Baire** if:

- For all topological spaces Ω
- For all continuous functions $\pi : \Omega \to \mathbb{R}^n$;

the preimage of A by π has the property of Baire in the space Ω.

Theorem

Assume $V = L$. Then every set $A \subseteq \mathbb{R}$ is the image of a universally Baire set by a continuous function

$$F : \mathbb{R} \to \mathbb{R}.$$

$L(A, \mathbb{R})$ where $A \subseteq \mathbb{R}$

Relativizing L to $A \subseteq \mathbb{R}$

Suppose $A \subseteq \mathbb{R}$. Define $L_\alpha(A, \mathbb{R})$ by induction on α by:

1. $L_0(A, \mathbb{R}) = V_{\omega+1} \cup \{A\}$,
2. (Successor case) $L_{\alpha+1}(A, \mathbb{R}) = \mathcal{P}_{\mathrm{Def}}(L_\alpha(A, \mathbb{R}))$,
3. (Limit case) $L_\alpha(A, \mathbb{R}) = \cup\{L_\beta(A, \mathbb{R}) \mid \beta < \alpha\}$.

- $L(A, \mathbb{R})$ is the class of all sets X such that $X \in L_\alpha(A, \mathbb{R})$ for some ordinal α.

The ultimate generalization of the projective sets

Theorem

Suppose that there is a proper class of Woodin cardinals.

(1) (Martin-Steel) *Suppose $A \subseteq \mathbb{R}$ is universally Baire.*
  - *Then A is determined.*

(2) (Steel) *Suppose $A \subseteq \mathbb{R} \times \mathbb{R}$ is universally Baire.*
  - *Then A has a choice function which is universally Baire.*

Theorem

Suppose that there is a proper class of Woodin cardinals and suppose $A \subseteq \mathbb{R}$ is universally Baire.

- *Then every set $B \in L(A, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ is universally Baire.*

- Thus $L(A, \mathbb{R}) \models \mathrm{AD}$, where AD is the Axiom of Determinacy.

Measuring the complexity of universally Baire sets

Definition

Suppose A and B are subsets of $\mathbb{R}$.

1. A is **weakly Wadge reducible** to B, $A \leq_{\text{Wadge}} B$, if there is a function $\pi : \mathbb{R} \to \mathbb{R}$ such that:
   - π is continuous on $\mathbb{R} \backslash \mathbb{Q}$.
   - Either $A = \pi^{-1}[B]$ or $A = \mathbb{R} \backslash \pi^{-1}[B]$.
2. A and B are **weakly Wadge bi-reducible** if
   - $A \leq_{\text{Wadge}} B$ and $B \leq_{\text{Wadge}} A$.
3. The **weak Wadge degree** of A is the equivalence class of all sets which are weakly Wadge bi-reducible with A.

An indication of deep structure

Theorem (Martin-Steel, Martin, Wadge)

Assume there is a proper class of Woodin cardinals. Then:

- *The weak Wadge degrees of the universally Baire sets are linearly ordered by weak Wadge reducibility.*
- *This linear order is a wellorder.*

Speculation

This structure really does suggest that the universally Baire sets can lead us to the ultimate generalization of the axiom $V = L$.

Gödel's transitive class HOD

Definition

A set M is a **transitive set** if $a \subset M$ for each $a \in M$.

- Each set V_α is a transitive set.
- Suppose M is a finite transitive set.
  - Then $M \in V_\omega$.
- V_ω is the union of all finite transitive sets.

Definition

HOD is the class of all sets X such that there exist $\alpha \in \mathrm{Ord}$ and $M \in V_\alpha$ such that

1. $X \in M$ and M is transitive.
2. Every element of M is definable in V_α from ordinal parameters.

$\mathrm{HOD}^{L(A,\mathbb{R})}$ and measurable cardinals

Definition

Suppose that $A \subseteq \mathbb{R}$. Then $\mathrm{HOD}^{L(A,\mathbb{R})}$ is the class HOD as defined within $L(A, \mathbb{R})$.

- The Axiom of Choice must hold in $\mathrm{HOD}^{L(A,\mathbb{R})}$
  - even if $L(A, \mathbb{R}) \models \mathrm{AD}$.

Theorem (Solovay:1967)

Suppose that $A \subseteq \mathbb{R}$ and $L(A, \mathbb{R}) \models \mathrm{AD}$.

- *Then ω_1^V is a measurable cardinal in $\mathrm{HOD}^{L(A,\mathbb{R})}$.*

The least measurable cardinal of $\mathrm{HOD}^{L(A,\mathbb{R})}$

Theorem

Suppose that there is a proper class of Woodin cardinals and that A is universally Baire.

- *Then ω_1^V is the least measurable cardinal in $\mathrm{HOD}^{L(A,\mathbb{R})}$.*

Definition

Suppose that $A \subseteq \mathbb{R}$ is universally Baire. Then

- $\Theta^{L(A,\mathbb{R})}$ is the supremum of the ordinals α such that there exists a surjection, $\pi : \mathbb{R} \to \alpha$, such that $\pi \in L(A, \mathbb{R})$.

$\mathrm{HOD}^{L(A,\mathbb{R})}$ and Woodin cardinals

Lemma

Suppose that there is a proper class of Woodin cardinals, A is universally Baire, and $\kappa \geq \Theta^{L(A,\mathbb{R})}$. *Then:*

- κ *is not a measurable cardinal in* $\mathrm{HOD}^{L(A,\mathbb{R})}$.

Theorem

Suppose that there is a proper class of Woodin cardinals and that A is universally Baire. Then:

- $\Theta^{L(A,\mathbb{R})}$ *is a Woodin cardinal in* $\mathrm{HOD}^{L(A,\mathbb{R})}$.

The axiom $V = \mathrm{Ultimate}\text{-}L$

A sentence φ is a Σ_2-sentence if it is of the form:

- *There exists an ordinal α such that $V_\alpha \models \psi$;*

for some sentence ψ.

The axiom for $V = \mathrm{Ultimate}\text{-}L$

- There is a proper class of Woodin cardinals.
- For each Σ_2-sentence φ, if φ holds in V then there is a universally Baire set $A \subseteq \mathbb{R}$ such that

$$\mathrm{HOD}^{L(A,\mathbb{R})} \models \varphi.$$

Some consequences of $V = \mathrm{Ultimate}\text{-}L$

Theorem ($V = \mathrm{Ultimate}\text{-}L$)

The Continuum Hypothesis holds.

Theorem ($V = \mathrm{Ultimate}\text{-}L$)

$V = \mathrm{HOD}$.

- Suppose there is a Cohen-blueprint for $V = \mathrm{Ultimate}\text{-}L$. Then:
  - The axiom $V = \mathrm{Ultimate}\text{-}L$ must hold and the blueprint is trivial.

Claim

Adopting the axiom $V = \mathrm{Ultimate}\text{-}L$ completely negates the ramifications of Cohen's method.

The language of large cardinals: elementary embeddings

Definition

Suppose X and Y are transitive sets. A function

$$j : X \to Y$$

is an **elementary embedding** if for all logical formulas $\varphi[x_0, \ldots, x_n]$ and all $a_0, \ldots, a_n \in X$,

- $(X, \in) \models \varphi[a_0, \ldots, a_n]$

if and only if

- $(Y, \in) \models \varphi[j(a_0), \ldots, j(a_n)]$

Extendible cardinals

Lemma

Suppose that $j : V_\alpha \to V_\beta$ is an elementary embedding. Then the following are equivalent.

(1) *j is not the identity.*

(2) *There is an ordinal $\eta < \alpha$ such that $j(\eta) \neq \eta$.*

- $\mathrm{CRT}(j)$ denotes the least ordinal η such that $j(\eta) \neq \eta$.

Definition (Reinhardt:(1974))

Suppose that δ is a cardinal.

- Then δ is an **extendible cardinal** if for each $\lambda > \delta$ there exists an elementary embedding

$$j : V_{\lambda+1} \to V_{j(\lambda)+1}$$

such that $\mathrm{CRT}(j) = \delta$ and $j(\delta) > \lambda$.

The transitive sets $H(\gamma)$

Definition

For each (infinite) cardinal γ:

- $H(\gamma)$ denotes the union of all transitive sets M such that $|M| < \gamma$.

- $H(\omega) = V_\omega$.
- $H(\omega_1)$ and $V_{\omega+1}$ are logically equivalent structures.
  - $H(\omega_1) \models \mathrm{ZF} \backslash \mathrm{Powerset}$
  - $H(\omega_1) \models$ "The Wellordering Principle".
- For any infinite cardinal γ,
  - $H(\gamma^+) \models \mathrm{ZF} \backslash \mathrm{Powerset}$.
  - $H(\gamma^+) \models$ "The Wellordering Principle".

Inner models

A transitive class is an **inner model** if

- $\mathrm{Ord} \subset M$,
- $M \models \mathrm{ZFC}$.

(meta) Lemma

Suppose M is a transitive class containing Ord. Then the following are equivalent:

- *M is an inner model.*
- *For each infinite cardinal γ,*

$$M \cap H(\gamma^+) \models \mathrm{ZFC} \backslash \mathrm{Powerset}.$$

- L and HOD are inner models.

The δ-cover and δ-approximation properties

Definition (Hamkins)

Suppose N is an inner model and that δ is an uncountable (regular) cardinal of V.

1. N has the δ-**cover property** if for all $\sigma \subset N$, if $|\sigma| < \delta$ then there exists $\tau \in N$ such that:
   - $\sigma \subset \tau$,
   - $|\tau| < \delta$.
2. N has the δ-**approximation property** if for all sets $X \subset N$, the following are equivalent.
   - $X \in N$.
   - For all $\sigma \in N$ if $|\sigma| < \delta$ then $\sigma \cap X \in N$.

- **Motivation:** If V is an extension of an inner model N by Cohen's method then
  - N has the δ-approximation property.
  - N has the δ-cover property;

  for **all** sufficiently large regular cardinals δ.

The uniqueness and universality theorems

Theorem (Hamkins)

Suppose N_0 and N_1 both have the δ-approximation property and the δ-cover property. Suppose

- $N_0 \cap H(\delta^+) = N_1 \cap H(\delta^+)$.

Then $N_0 = N_1$.

Theorem (Hamkins)

Suppose that N is an inner model with the δ-cover and δ-approximation properties, $\kappa > \delta$, and that κ is an extendible cardinal.

- *Then κ is an extendible cardinal in N.*

- The theorem **holds** for all the large cardinal notions on the earlier list except the last one, Axiom I_0 cardinals.
  - These are the **universality theorems**.
- The theorem **fails** in the case that κ is an Axiom I_0 cardinal.

The δ-genericity property and strong universality

- Suppose N is an inner model and that $\sigma \subset N$. Then $N[\sigma]$ denotes the smallest inner model M such that
  - $N \subseteq M$ and $\sigma \in M$.

Definition

Suppose that N is an inner model and δ is strongly inaccessible.

- Then N has the δ-**genericity property** if for all $\sigma \subseteq \delta$, if $|\sigma| < \delta$ then
  - $N[\sigma] \cap V_\delta$ is a Cohen extension of $N \cap V_\delta$:

Theorem

Suppose that:

- *N has the δ-approximation property, the δ-cover property,* **and** *the δ-genericity property.*

Suppose that the $\mathrm{Axiom\ I}_0$ *holds at λ, for a proper class of λ.*

- *Then in N, the* $\mathrm{Axiom\ I}_0$ *holds at λ, for a proper class of λ.*

The Ultimate-L Conjecture

Ultimate-L Conjecture

Suppose that δ is an extendible cardinal. Then **provably** there is an inner model N such that:

1. N has the δ-cover and δ-approximation properties.
2. N has the δ-genericity property.
3. $N \models$ "$V = \mathrm{Ultimate}\text{-}L$".

- The $\mathrm{Ultimate}$-L Conjecture is an existential number theoretic statement.
  - If it is undecidable then it must be **false**.

Claim

The $\mathrm{Ultimate}$-L Conjecture **must** be either true or false

- it cannot be meaningless.

Set Theory faces one of two futures

- The $\mathrm{Ultimate}\text{-}L$ Conjecture reduces the **entire** post-Cohen debate on Set Theoretic truth to a single question which
  - **must** have an answer.

Future 1: The $\mathrm{Ultimate}\text{-}L$ Conjecture is true.

- Then the axiom $V = \mathrm{Ultimate}\text{-}L$ is very likely the key missing axiom for V.
  - There is **no** generalization of Scott's Theorem for the axiom $V = \mathrm{Ultimate}\text{-}L$.
  - **All** the questions which have been shown to be unsolvable by Cohen's method are resolved modulo large cardinal axioms.

Future 2: The $\mathrm{Ultimate}\text{-}L$ Conjecture is false.

- Then the program to understand V by generalizing the success in understanding $V_{\omega+1}$ and the projective sets, fails.

- Which is it?