A new characterisation of virtually free groups.

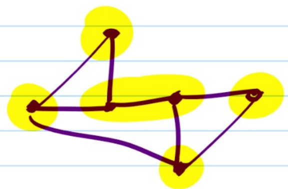
Γ - graph $\qquad$ Δ - finite graph

$V(\Delta) = \{v_1, \dots, v_n\}$ $\qquad$ (vertices)

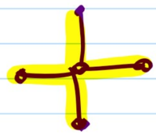
Def.

Δ is a *minor* of Γ if $\exists$ finite connected disjoint subsets $V_1, \dots, V_n$ of $V(\Gamma)$ s.t. $\exists$ edge between some vertex of V_i and some vertex of V_j if $(v_i, v_j) \in E(\Delta)$.

Γ



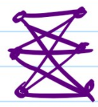
Δ



V_i = "branch sets" of Δ in Γ $\qquad$ $\Delta \prec \Gamma$

Th'm (Kuratowski, 1930)

A graph Γ is planar $\iff$ $K_5, K_{3,3} \nprec \Gamma$



excluded minor $\rightsquigarrow$ "restriction".

Def.

Γ is *minor-excluded* if $\exists$ finite graph Δ s.t. $\Delta \nprec \Gamma$ $\quad$ ($\iff \exists\, m$ s.t. $K_m \nprec \Gamma$)

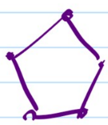
Th'm (Ostrovskii - Rosenthal, 2014)

Γ loc. finite, connected, $K_m \nprec \Gamma \implies \operatorname{asdim}\Gamma \le 4^m - 1$

What about groups?

Planarity

eg. C_5



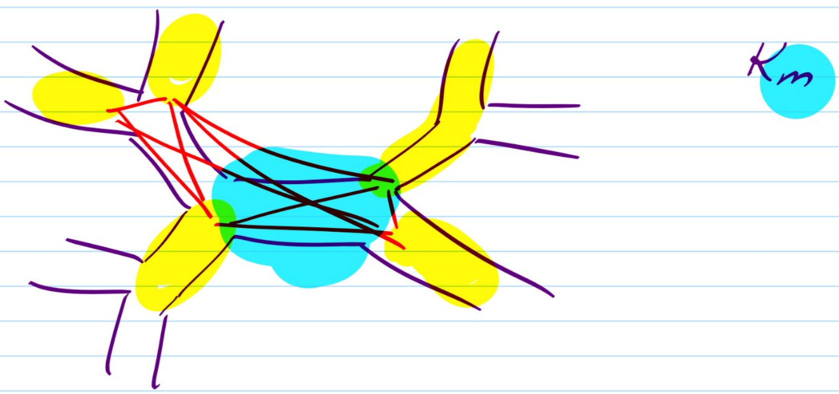
Th'm (O.-R.)

Minor exclusion depends on the gen. set (eg $\mathbb{Z}^2$)

... $\exists\, S \dots$? $\qquad$ $\forall\, S \dots$?

Th'm (O.-R.)

Virtually free $\implies$ minor excluded wrt. all finite gen. sets.



Question: Is the converse true?

Virtually free $\iff$ quasi-isom. to tree

$\iff$ fin. pres. and asdim $= 1$

$\iff$ context-free word problem ...

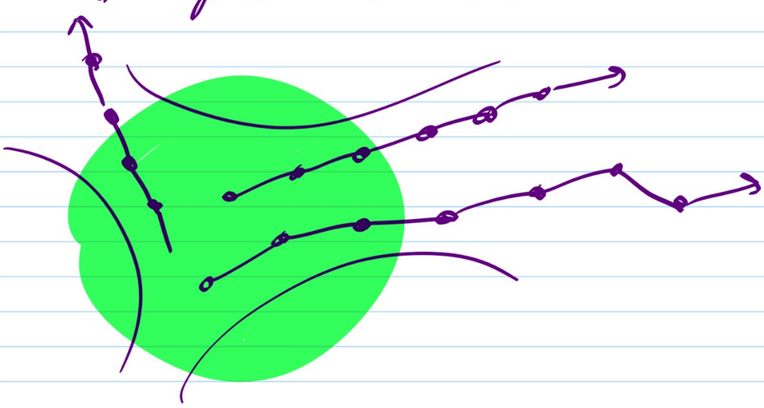
- many use Dunwoody's Accessibility result.

Ends and accessibility.

ray :

"one-way ∞ path".

Two rays in Γ are equivalent if they stay in the same connected component of $\Gamma \setminus F$ $\forall$ finite $F \subset \Gamma$.



The equivalence classes are the *ends* of Γ.

The *no. of ends* $e(G)$ is a Q.-I invariant for groups.

Th'm

$e(G) \in \{0, 1, 2, \infty\}$.

$e(G) > 1 \iff G = A *_C B$ or $A *_C$

"can decompose G" (C finite)

$$G = A *_C B$$

$$(D * E) * (\cdots)$$

G <u>accessible</u> if the process of decomposition terminates in a finite no. of steps.

<u>Th'm</u> (Dunwoody, 1985)

G finitely presented $\Rightarrow$ G accessible.

<u>Th'm</u> (O.-R.)

- $e(G) = 0 \Rightarrow \mathrm{Cay}(G, S)$ minor excl. $\forall S$
- $e(G) = 1 \Rightarrow \exists\, S$ s.t. $\mathrm{Cay}(G, S)$ is not minor excluded. (✩)
- $e(G) = 2$ ($\Leftrightarrow G$ virtually $\mathbb{Z}$) $\Rightarrow \mathrm{Cay}(G, S)$ minor excl. $\forall S$.

<u>Q:</u> $e(G) = \infty \ldots$?

<u>Characterising virtually free groups.</u>

<u>Th'm</u> (K., 2020)

If G is a f.g. group that is minor excl. w.r.t. $\forall S$, then G is virtually free.

<u>Prop.</u> True if G is accessible.

<u>Proof.</u>

If G accessible,

$$G$$

$$A *_C B$$

$$\vdots$$

$$e \leq 1$$

By (✩), we cannot have subgroups with $e = 1$.

So all groups at the end of decomp. must have $e = 0$, so all finite

– then (by Karass – Pietrowski – Solitar, 1973) G virt. free.

Th'm

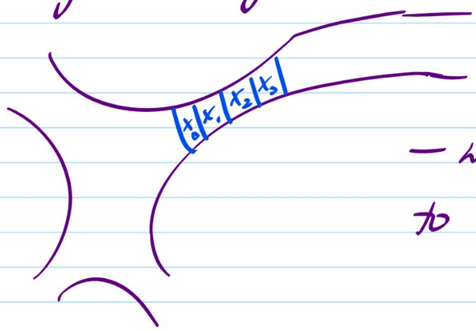
G minor excl. $\forall S \implies G$ accessible.

Sketch:

Thomassen-Woess (1991): characterisation of access.

Let $\langle S \rangle = G$.

G is not accessible $\iff \forall m > 0$, $Cay(G,S)$ has a "thin" end s.t. the maximal no. of disjoint rays in this end is $\geq m$.



— we can use the m rays to construct a K_m minor

$(S \leadsto S \cup S^2 \cup S^3)$

Remark.

$\exists\, G, S$ s.t. $e(G) = \infty$ and $Cay(G,S)$ not minor excl.

Q:

Characterize groups admitting at least one gen. set w.r.t. which $Cay(G)$ is minor excl.

cf. Bonamy et al.: $\text{asdim}(G) \leq 2$.