Geometric property (T) for non-discrete spaces

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Geometric property (T) 1/3

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We can consider T as a matrix.

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We can consider T as a matrix.

Definition (controlled support)

T has *controlled support* if $\sup(d(x,y) \mid T_{x,y} \neq 0) < \infty$.

In particular, T acts on each $L^2 G_n$.

Geometric property (T) 2/3

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Always $0 \in \sigma(\Delta)$ and $\Delta \geq 0$. The graph G is an expander iff $\sigma(\Delta) \subseteq \{0\} \cup [\gamma, \infty)$ for some $\gamma > 0$.

Geometric property (T) 3/3

Definition (representation)

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Definition (Willett, Yu 2010: Geometric property (T))

G has geometric property (T): there is $\gamma > 0$ with $\sigma_{\max}(\Delta) \subseteq \{0\} \cup [\gamma, \infty)$.

Why (T)?

Example (box space)

Let G a group, normal subgroups $G_1 \supseteq G_2 \supseteq \cdots$ with G/G_n finite and $\bigcap_n G_n = \{1\}$.
Finitely generated by S.
Box space is union of Cayley graphs G/G_n.

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Theorem [Willett-Yu]

G has property (T) iff $\bigsqcup_n G/G_n$ has geometric property (T).

Coarse equivalence 1/3

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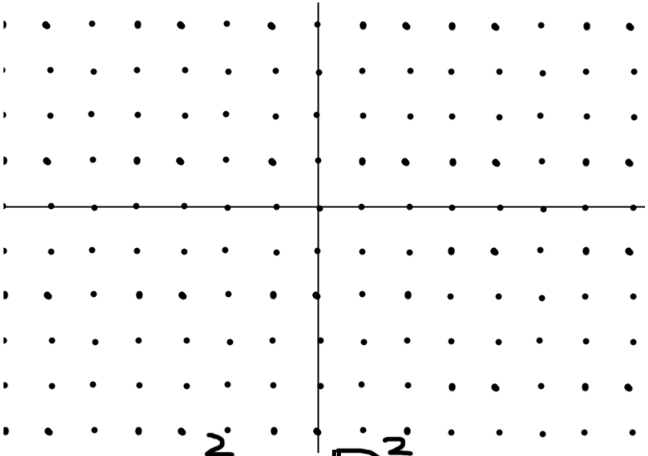
Concretely: there are $f\colon X \to Y$ and $g\colon Y \to X$ such that

$$\begin{aligned} d(f(x), f(x')) &\leq \rho(d(x, x')) \\ d(g(y), g(y')) &\leq \rho(d(y, y')) \\ d(fg(y), y) &\leq C \\ d(gf(x), x) &\leq C. \end{aligned}$$

Coarse equivalence 2/3



$\mathbb{Z}^2 \approx \mathbb{R}^2$

Coarse equivalence 3/3

Theorem [Willett-Yu]

Geometric property (T) is coarse invariant:
if (G_n) and (H_n) are coarsely equivalent graph sequences, then (G_n) has geometric (T) iff (H_n) has geometric (T).

Generalization

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- (sequences of) Riemannian manifolds
- Warped systems

Bounded geometry 1/2

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Definition

Bounded geometry for metric space: there is R, s.t. for every S, there is an N, such that:
every S-ball is covered by at most N different R-balls.

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Theorem [W.]

X has bounded geometry iff there is a measure μ that is *uniformly bounded* and *gordo*.

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Uniformly bounded

For every R there is a C such that every R-ball has volume at most C.

Gordo

There is an R and $\epsilon > 0$ such that every R-ball has volume at least ϵ.

Operators with controlled support

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if T and S have controlled support:
so do $T + S$, TS and T^*.

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Example

Canonical representation: take $\mathcal{H} = L^2X$ and $\rho(T) = T$.

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Definition (maximal norm)

$$\|T\|_{\max} = \sup_{(\rho,\mathcal{H})} \|\rho(T)\| .$$

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Definition (Roe algebra)

Roe algebra $C^*_{\max}(X)$ is completion of $\mathbb{C}_{\text{cs}}[X]$ w.r.t. maximal norm.

Laplacians

Definition (Laplacian)

Let

$$\Delta_R \colon L^2X \to L^2X$$
$$\Delta_R\xi(x) = \int_{d(y,x)\leq R} (\xi(x) - \xi(y))d\mu(y).$$

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If $R \geq S$ then $\Delta_R \geq \Delta_S \geq 0$.

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Example

if $\mathcal{H} = L^2 X$, constant vectors are the functions that are constant on each component.

Geometric (T)

Definition (geometric (T)) [W 2020]

X has geometric property (T) if there is R such that:

- for every representation $(\rho, \mathcal{H})$, we have $\mathcal{H}_c = \ker(\rho(\Delta_R))$
- there is $\gamma > 0$ such that $\sigma_{\max}(\Delta_R) \subseteq \{0\} \cup [\gamma, \infty)$.

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Theorem (coarse invariance)

if X and Y are coarsely equivalent, X has (T) iff Y has (T).

Theorem (connected spaces)

if X is connected, then X has (T) if and only if X is either bounded or not amenable.

Manifolds

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Assume: injectivity radius positive, Ricci curvature bounded below (by negative constant)

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Theorem [W 2020]

Property (T) iff there is γ such that the spectrum of $\rho(\Delta_M)$ is contained in $\{0\} \cup [\gamma, \infty]$ for every representation $(\rho, \mathcal{H})$.

Warped systems

Let M a compact Riemannian manifold. It has a metric and a measure.
Let Γ a group generated by finite S, with m.p. action $\alpha\colon \Gamma \curvearrowright M$.

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For positive t, we make a new metric on M: largest metric d_t s.t. $d_t(x, y)) \leq td(x, y)$ and $d_t(x, s \cdot x) \leq 1$.

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Union $\bigsqcup_{t \in \mathbb{N}} M \times \{t\}$ with metric d_t on $M \times \{t\}$.

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Question: when geometric property (T)?

Possible answer: if Γ has property (T) and the action is ergodic.