New perspectives in hermitian K-theory II

Yonatan Harpaz

CNRS, Université Paris 13

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Joint work with Baptiste Calmès, Emanuele Dotto, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus and Wolfgang Steimle.

Recollection: Poincaré ∞-categories

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- A *hermitian ∞-category* is a stable ∞-category $\mathcal{C}$ equipped with a hermitian structure, that is a quadratic (reduced and 2-excisive) functor $\mathfrak{Q}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p$.

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- $(\mathcal{C}, \Omega)$ is *Poincaré* if there exists an equivalence of ∞-categories $\mathrm{D}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ together with a natural equivalence

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- $\mathrm{D} = \mathrm{D}_{\Omega}$ is essentially uniquely determined by Ω, and is called the *duality* associated to Ω.

The category of Poincaré ∞-categories

Any hermitian functor $(f, \eta)\colon (\mathcal{C}, \Omega) \to (\mathcal{C}', \Omega')$ between Poincaré ∞-categories determines a natural transformation

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Proposition

The ∞-category $\mathrm{Cat}^{\mathrm{p}}_\infty$ admits small limits and colimits, and the forgetful functor

$$\mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{Cat}^{\mathrm{ex}}_\infty$$

preserves small limits and colimits, where $\mathrm{Cat}^{\mathrm{ex}}_\infty$ is the ∞-category of stable ∞-categories and exact functors.

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Non-example

The functor $\mathrm{Pn}(-)$ is product preserving but not group-like.

Poincaré-Verdier sequences

Definition

A sequence of Poincaré functors

$$(\mathcal{C}, \Upsilon) \xrightarrow{(f,\eta)} (\mathcal{D}, \Phi) \xrightarrow{(p,\vartheta)} (\mathcal{E}, \Psi)$$

with vanishing composite is called a *Poincaré-Verdier sequence* if it is both a fiber and a cofiber sequence in $\mathrm{Cat}^{\mathrm{p}}_{\infty}$. The sequence is said to *split* if p admits both a left and a right adjoint.

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- A functor $(f, \eta): (\mathcal{C}, \Omega) \to (\mathcal{D}, \Phi)$ is a split Poincaré-Verdier inclusion if and only if f is fully-faithful, admits a right adjoint, and $\eta: \Omega \Rightarrow f^*\Omega'$ is an equivalence.

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- A functor $(p,\vartheta): (\mathcal{D}, \Phi) \to (\mathcal{E}, \Psi)$ is a split Poincaré-Verdier projection if and only if p admits a fully-faithful left adjoint $g: \mathcal{E} \to \mathcal{D}$ and the composed map $g^*\Psi \xRightarrow{g^*\vartheta} g^*p^*\Phi \Longrightarrow \Phi$ is an equivalence.

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The metabolic sequence

There is a canonical split Poincaré-Verdier sequence

$$\begin{array}{ccccc} (\mathcal{C}, Ϙ^{[-1]}) & \longrightarrow & \mathrm{Met}(\mathcal{C}, Ϙ) & \longrightarrow & (\mathcal{C}, Ϙ) \\ & & [L \to X] & \longmapsto & X \end{array}$$

Poincaré-Verdier squares

Definition

A commutative square of Poincaré ∞-categories

$$\begin{array}{ccc} (\mathcal{C}, \Omega) & \longrightarrow & (\mathcal{D}, \Phi) \\ \downarrow & & \downarrow \\ (\mathcal{C}', \Omega') & \longrightarrow & (\mathcal{D}', \Phi') \end{array}$$

is called a (split) Poincaré-Verdier square if it is cartesian and its vertical legs are (split) Poincaré-Verdier projections.

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Let $\mathcal{E}$ be an ∞-category with finite limits and $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ a functor which preserves final objects. We say that $\mathcal{F}$ is *Verdier-localizing* if it sends Poincaré-Verdier squares to fiber squares, and *additive* if it sends split Poincaré-Verdier squares to fiber squares.

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Example

The functor $\mathrm{Pn}\colon \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathcal{S}$ is Verdier-localizing.

More examples

In the diamond diagram

$$\begin{array}{ccccc} & & (\mathcal{C}, Ϙ) & & \\ & & \downarrow s & \overset{\mathrm{id}}{\searrow} & \\ \mathrm{Met}(\mathcal{C}, Ϙ) & \longrightarrow & \mathrm{Q}_1(\mathcal{C}, Ϙ) & \xrightarrow{d_1} & (\mathcal{C}, Ϙ) \\ & \searrow & \downarrow & & \\ & & \mathrm{Hyp}(\mathcal{C}) & & \end{array}$$

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both the vertical and the horizontal sequence are split Poincaré-Verdier sequences. Here the bottom vertical functor sends $[X \xleftarrow{\alpha} W \xrightarrow{\beta} X'] \in \mathrm{Q}_1(\mathcal{C})$ to $(\mathrm{fib}[\alpha], \mathrm{D}\,\mathrm{cof}[\beta]) \in \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$.

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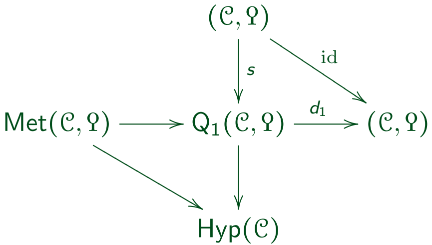

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Corollary

If $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ is a group-like additive functor then

$$\mathcal{F}(Q_1(\mathcal{C}, \Omega)) \simeq \mathcal{F}(\mathcal{C}, \Omega) \times \mathcal{F}(\mathrm{Met}(\mathcal{C}, \Omega)) \simeq \mathcal{F}(\mathcal{C}, \Omega) \times \mathcal{F}(\mathrm{Hyp}(\mathcal{C}))$$

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The fibration theorem

If $(\mathcal{D}, \Phi) \to (\mathcal{E}, \Psi)$ is a split Poincaré-Verdier projection then the induced functor

$$p_* : \mathrm{Cob}(\mathcal{D}, \Phi) \to \mathrm{Cob}(\mathcal{E}, \Psi)$$

is a bicartesian fibration.

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in $\mathrm{Cat}^{\mathrm{b}}(\mathcal{E}, \Psi)$ induces an adjunction between the fibers of p_* over (X, q) and (X', q'). Since every adjunction induces a homotopy equivalence on realizations, the family of geometric realizations of fibers of p_* is constant up to homotopy over $\mathrm{Cob}(\mathcal{E}, \Psi)$. It hence also coincides with the homotopy fibers of the induced map $|\mathrm{Cob}(\mathcal{D}, \Phi)| \to |\mathrm{Cob}(\mathcal{E}, \Psi)|$.

Corollary

The functors $(\mathcal{C}, \Omega) \mapsto |\mathrm{Cob}(\mathcal{C}, \Omega)|$ and $(\mathcal{C}, \Omega) \mapsto \mathcal{GW}(\mathcal{C}, \Omega)$ are additive.

Proving the fibration theorem

To given an idea of the proof, let us just indicate how one can find p_*-(co)cartesian edges in $\mathrm{Cob}(\mathcal{D}, \Phi)$. For this, we note that the fibration theorem has a non-hermitian precursor:

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Let $p: \mathcal{D} \to \mathcal{E}$ be an exact functor which admits fully-faithful left and right adjoints. Then the functor $p_*: \mathrm{Span}(\mathcal{D}) \to \mathrm{Span}(\mathcal{E})$ is a bicartesian fibration.

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Furthermore, a span $X \xleftarrow{\alpha} W \xrightarrow{\beta} Y$ in $\mathcal{D}$ is p_*-cocartesian if and only if α is p-cartesian and β is p-cocartesian, and is p_*-cartesian if this statement holds with the role of α and β reversed.

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Lemma

Let $(p, \vartheta)\colon (\mathcal{D}, \Phi) \to (\mathcal{E}, \Psi)$ be a split Poincaré-Verdier projection and let $\beta\colon W \to Y$ be a p-cocartesian edge in $\mathcal{D}$. Then the map $\Phi(Y) \to \Phi(W) \times_{\Psi(p(W))} \Psi(p(Y))$ is an equivalence.

Proving the fibration theorem

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Proof.

Let $g\colon \mathcal{E} \to \mathcal{D}$ be a fully-faithful left adjoint to p. Then an arrow $\beta\colon W \to Y$ is p-cocartesian if and only if the square

$$\begin{array}{ccc} gp(W) & \xrightarrow{gp(\beta)} & gp(Y) \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu} \\ W & \xrightarrow{\beta} & Y \end{array}$$

is exact. By the identification $\Phi(g(-)) \simeq \Psi(-)$ it will suffice to show that Φ sends the above square to an exact square of spectra. Since Φ is quadratic, the obstruction to this is the spectrum

$$\mathrm{B}_\Phi(\mathrm{cof}[gp(\beta)], \mathrm{cof}[\nu]) = \hom_{\mathcal{D}}(\mathrm{cof}[gp(\beta)], \mathrm{D}_\Phi \,\mathrm{cof}[\nu])$$
$$= \hom_{\mathcal{E}}(\mathrm{cof}[p(\beta)], \mathrm{D}_\Psi \,\mathrm{cof}[p(\nu)]) \qquad = 0$$

□

Proving the fibration theorem

Given a Poincaré object (X, q) in $(\mathcal{D}, \Phi)$ and a cobordism

$$(p(X), p(q)) \xleftarrow{\alpha} (W, \eta) \xrightarrow{\beta} (Y, r)$$

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- The unimoduarity of (p, η', r') can then be deduced from the unicity of cartesian and cocartesian lifts.

Consequences of additivity

Summary

The functor $\mathrm{Cob}(-)$ sends split Poincaré-Verdier sequences to bicartesian fibrations $\Rightarrow$ the functor $|\mathrm{Cob}(-)|$ is additive $\Rightarrow$ the functor $\mathcal{GW}(-)$ is additive.

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Karoubi's conjecture

Karoubi's fundamental theorem for Grothendieck-Witt spaces

There is a natural equivalence

$$\mathcal{V}(\mathcal{C}, \Omega) \simeq \Omega\mathcal{U}(\mathcal{C}, \Omega^{[2]})$$

For R a ring and M an invertible module with involution over R one has natural equivalences of Poincaré ∞-categories, yielding four types of fundamental theorems for rings:

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When $\frac{1}{2} \in R$ these claims are all equivalent to the classical Kaourbi's fundamental theorem.

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When 2 is not invertible, the last two statements apply to classical $\mathcal{U}$- and $\mathcal{V}$-theory of rings.

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Karoubi's fundamental theorem for Grothendieck-Witt spaces

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$$\mathcal{V}(\mathcal{C}, Ϙ) \simeq \Omega\mathcal{U}(\mathcal{C}, Ϙ^{[2]})$$

For R a ring and M an invertible module with involution over R one has natural equivalences of Poincaré ∞-categories, yielding four types of fundamental theorems for rings:

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When $\frac{1}{2} \in R$ these claims are all equivalent to the classical Kaourbi's fundamental theorem.

When 2 is not invertible, the last two statements apply to classical $\mathcal{U}$- and $\mathcal{V}$-theory of rings. They give a generalization of Karoubi's fundamental theorem conjectured by Karoubi and Giffen.

More consequences of additivity

Definition

For $(\mathcal{C}, \Omega)$ Poincaré, a full stable subcategory $\mathcal{A} \subseteq \mathcal{C}$ is called *isotropic* if

- Ω vanishes when restricted to $\mathcal{A}$.
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Example

- $\mathrm{Met}(\mathcal{C}, \Omega)$ admits a Lagrangian consisting of the equivalences $L \xrightarrow{\simeq} X$
- $\mathrm{Q}_1(\mathcal{C}, \Omega)$ admits an isotropic subcategory consisting of the spans of the form $0 \leftarrow W \xrightarrow{\simeq} X$. Its homology is canonically equivalence to $(\mathcal{C}, \Omega)$.
- $\mathrm{Q}_n(\mathcal{C}, \Omega)$ admits an isotropic subcategory with homology $\mathrm{Q}_{n-1}(\mathcal{C}, \Omega)$ consisting of the sequence of spans $0 \leftarrow 0 \rightarrow 0 \leftarrow \ldots \rightarrow 0 \leftarrow W \xrightarrow{\simeq} X$.

Isotropic decomposition

Proposition

Let $(\mathcal{C}, \mathfrak{Q})$ *be a Poincaré* ∞*-categorywith an isotropic subcategory* $\mathcal{A} \subseteq \mathcal{C}$*. Then for a group-like additive functor* $\mathcal{F} \colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ *there is a canonical decomposition*

$$\mathcal{F}(\mathcal{C}, \mathfrak{Q}) \simeq \mathcal{F}(\mathsf{Hlgy}(\mathcal{A})) \times \mathcal{F}(\mathsf{Hyp}(\mathcal{A})).$$

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Corollary

If $(\mathcal{C}, \mathfrak{Q})$ is a Poincaré ∞-category with isotropic subcategory $\mathcal{A} \subseteq \mathcal{C}$ then

$$\mathcal{GW}(\mathcal{C}, \mathfrak{Q}) \simeq \mathcal{GW}(\mathsf{Hlgy}(\mathcal{A})) \times \mathcal{K}(\mathcal{A}).$$

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Example

For every group-like additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ we have

$$\mathcal{F}(\mathsf{Q}_n(\mathcal{C}, \Omega)) \simeq \mathcal{F}(\mathcal{C}, \Omega) \times \mathcal{F}(\mathsf{Hyp}(\mathcal{C}))^n.$$

In particular

$$\mathcal{GW}(\mathsf{Q}_n(\mathcal{C}, \Omega)) \simeq \mathcal{GW}(\mathcal{C}, \Omega) \times \mathcal{K}(\mathcal{C})^n.$$

Example: additivity of K-theory

Let $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a fiber sequence of stable ∞-categories such that p admits a fully-faithful right adjoint $r\colon \mathcal{E} \to \mathcal{D}$.

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$\Rightarrow$ the Poincaré ∞-category $\mathrm{Hyp}(\mathcal{D})$ admits a Lagrangian inclusion

$$(i, r^{\mathrm{op}})\colon \mathcal{C} \times \mathcal{E}^{\mathrm{op}} \to \mathcal{D} \times \mathcal{D}^{\mathrm{op}}.$$

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If $\mathcal{F}: \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ is a group-like additive functor then

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Corollary (Waldhausen additivity)

$$\mathcal{K}(\mathcal{D}) \simeq \mathcal{K}(\mathcal{C}) \times \mathcal{K}(\mathcal{E}).$$

Towards a Grothendieck-Witt spectrum

Extending the perspective:

Towards a Grothendieck-Witt spectrum

Extending the perspective:

Cobordisms categories with coefficients

Let $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ be an additive functor and $(\mathcal{C}, \Qoppa)$ a Poincaré ∞-category. Then the simplicial space $\mathcal{F}\,\mathrm{Q}_{\bullet}(\mathcal{C}, \Qoppa^{[1]})$ is a Segal space. Define $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Qoppa)$ to be the associated ∞-category.

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- $\mathcal{F}\,\mathrm{Q}_{\bullet}(\mathcal{C}, \Omega^{[1]})$ is complete if $\mathcal{F}$ is limit preserving, but not in general.

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- If $\mathcal{F}$ is additive and group-like then by isotropic decomposition

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  is not just a Segal object, but a *groupoid object*, which is the action groupoid (known also as the bar construction) associated to the translation action of $\mathcal{F}(\mathsf{Hyp}(\mathcal{C}))$ on $\mathcal{F}(\mathcal{C}, \mathfrak{P}^{[1]})$ via the map induced by the Poincaré functor $\mathsf{Hyp}(\mathcal{C}) \to (\mathcal{C}, \mathfrak{P}^{[1]})$ sending $(X, Y) \mapsto X \oplus \mathrm{D}_{\mathfrak{P}^{[1]}} Y$.

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Generalized additivity

Cobordisms categories with coefficients

Let $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ be an additive functor and $(\mathcal{C}, \Omega)$ a Poincaré ∞-category. Then the simplicial space $\mathcal{F}\,\mathrm{Q}_{\bullet}(\mathcal{C}, \Omega^{[1]})$ is a Segal space. Define $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Omega)$ to be the associated ∞-category.

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The generalized fibration theorem

For every additive $\mathcal{F}\colon \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathcal{S}$ *the functor* $\mathrm{Cob}^{\mathcal{F}}(-)$ *sends split Poincaré-Verdier projections to bicartesian fibrations.*

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Corollary

For every additive $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ the functor $|\mathrm{Cob}^{\mathcal{F}}(-)|$ is additive.

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Corollary

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$\Rightarrow$ We can iterate this construction.

For $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ additive the commutative square

$$\begin{array}{ccc} X & \longmapsto & [0 \leftarrow X \rightarrow 0] \\ \cap & & \cap \\ (\mathcal{C}, \Omega) & \longrightarrow & \mathrm{Q}_1(\mathcal{C}, \Omega^{[1]}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & (\mathcal{C}, \Omega^{[1]}) \times (\mathcal{C}, \Omega^{[1]}) \end{array}$$

induces a natural map

$$\mathcal{F}(\mathcal{C}, \Omega) \to \Omega|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \Omega)|.$$

From spaces to spectra

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Lemma

Let $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ be an additive functor. If $\mathcal{F}$ is group-like then $|\mathrm{Cob}^{\mathcal{F}}(-)|$ is group-like and $\mathcal{F} \Rightarrow \Omega|\mathrm{Cob}^{\mathcal{F}}(-)|$ is an equivalence.

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Let $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ *be an additive functor. If* $\mathcal{F}$ *is group-like then* $|\mathrm{Cob}^{\mathcal{F}}(-)|$ *is group-like and* $\mathcal{F} \Rightarrow \Omega|\mathrm{Cob}^{\mathcal{F}}(-)|$ *is an equivalence.*

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Proof.

- Since $\mathcal{F}$ is group-like $\mathcal{F}\,\mathsf{Q}_{\bullet}(\mathcal{C}, \mathfrak{P}^{[1]})$ is the action groupoid of $\mathcal{F}(\mathsf{Hyp}(\mathcal{C}))$ acting on $\mathcal{F}(\mathcal{C}, \mathfrak{P}^{[1]})$, and its geometric realization is hence a model for the cofiber of $\mathcal{F}(\mathsf{Hyp}(\mathcal{C})) \to \mathcal{F}(\mathcal{C}, \mathfrak{P}^{[1]})$ in E_{∞}-groups.

□

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- Since $\mathcal{F}$ is additive and group-like we have $\mathcal{F}(\mathsf{Hyp}(\mathcal{C})) \simeq \mathcal{F}(\mathsf{Met}(\mathcal{C}, \Omega^{[1]}))$ and hence we can identify this cofiber with the cofiber of the map $\mathcal{F}(\mathsf{Met}(\mathcal{C}, \Omega^{[1]})) \to \mathcal{F}(\mathcal{C}, \Omega^{[1]})$.

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Lemma

Let $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ *be an additive functor. If* $\mathcal{F}$ *is group-like then* $|\mathrm{Cob}^{\mathcal{F}}(-)|$ *is group-like and* $\mathcal{F} \Rightarrow \Omega|\mathrm{Cob}^{\mathcal{F}}(-)|$ *is an equivalence.*

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- This follows from the additivity of $\mathcal{F}$ applied in the case of metabolic sequence of $(\mathcal{C}, \Omega^{[1]})$.

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Delooping group-like additive functors

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For $\mathcal{F}$ additive and group-like we have an equivalence $\mathcal{F} \stackrel{\simeq}{\Rightarrow} \Omega|\mathrm{Cob}^{\mathcal{F}}(-)|$.

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with $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_n(\mathcal{C}, Ϙ) := |\mathrm{Cob}^{\mathcal{F}_{n-1}}(\mathcal{C}, Ϙ)|$ for $n \geq 1$. Structure maps $\mathcal{F}_n(\mathcal{C}, Ϙ) \xrightarrow{\simeq} \Omega\mathcal{F}_{n+1}(\mathcal{C}, Ϙ)$ given as above.

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Observation

For any group-like additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ we have that $\mathrm{Cob}^{\mathcal{F}}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ is additive and $\Omega^{\infty}\mathrm{Cob}^{\mathcal{F}} \simeq \mathcal{F}$.

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The Grothendieck-Witt spectrum

Definition

We define the *Grothendieck-Witt spectrum* functor $\mathrm{GW}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ by $\mathrm{GW}(\mathcal{C}, \mathfrak{Q}) := \mathbb{C}\mathrm{ob}^{\mathcal{GW}}(\mathcal{C}, \mathfrak{Q})$.

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Example

For the hyperbolic Poincaré ∞-category $\mathsf{Hyp}(\mathcal{C})$ one has a canonical equivalence $\mathsf{GW}(\mathsf{Hyp}(\mathcal{C})) \simeq \mathsf{K}(\mathcal{C})$.

Spectral Bott-Genauer sequence

Applying additivity in the case of the metabolic sequence yields

The Bott-Genauer sequence (spectral version)

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Corollary

The negative homotopy groups of $\mathrm{GW}(\mathcal{C}, \Omega)$ are the negative L-groups.

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Theorem (Universality for $\mathcal{GW}$)

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Define Poincaré ∞-categories $\mathrm{Null}_n(\mathcal{C}, \Omega)$ as the fiber of

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where $i_0 \colon [0] \to [n]$ has image $\{0\}$. One obtains a square of simplicial Poincaré ∞-categories

$$\begin{array}{ccc} (\mathcal{C}, \Omega) & \longrightarrow & \mathrm{Null}_\bullet(\mathcal{C}, \Omega^{[1]}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Q}_\bullet(\mathcal{C}, \Omega^{[1]}) \end{array}$$

where the right vertical map is restricted from the face maps $d_0 \colon \mathrm{Q}_{\bullet+1}(\mathcal{C}, \Omega^{[1]}) \to \mathrm{Q}_\bullet(\mathcal{C}, \Omega^{[1]})$.

Proof of the key statement

Goal: show that if $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ is additive then

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is cocartesian in $\mathsf{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S})$. Here the top right corner is contractible since $\mathrm{Cob}^{\mathcal{F}}_{0/}(\mathcal{C}, \mathfrak{P})$ has an initial object.

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The operation $\mathcal{F} \mapsto \mathcal{F}\,\mathsf{Q}_n(-)$ has a right adjoint $\mathcal{F} \mapsto \mathcal{F}\,\mathsf{Q}^n(-)$, given by the *dual* Q*-construction*.

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- $\mathrm{Q}^1(\mathcal{C}, \Omega)$ is the Poincaré ∞-category whose objects are *cospans* with hermitian structure given by $\Omega_1([X \to W \leftarrow X'] = \Omega(X) \coprod_{\Omega(W)} \Omega(X')$.

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- Similarly, $\mathrm{Q}^n(\mathcal{C}, Ϙ)$ can be described as the ∞-category of diagrams encoding sequences of n composable cospans with hermitian structure given by a colimit on the diagram.
- There is also a dual to $\mathrm{Null}_n(\mathcal{C}, Ϙ)$ which sits in a split Poincaré-Verdier sequence of the form

$$(\mathcal{C}, Ϙ) = \mathrm{Q}^0(\mathcal{C}, \mathrm{Q}) \to \mathrm{Q}^{n+1}(\mathcal{C}, Ϙ) \to \mathrm{Null}^n(\mathcal{C}, Ϙ)$$

End of proof

Mapping into a test object $\mathcal{G}$ and using adjunction, it will suffice to show that for $\mathcal{G}$ additive the square

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is a fiber square of additive functors. Since totalizations preserve pullback fiber squares we can check this levelwise.

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$$\mathcal{G}(\mathsf{Q}^n(\mathcal{C},\mathcal{Q}^{[-1]})) \to \mathcal{G}(\mathsf{Null}^n(\mathcal{C},\mathcal{Q}^{[-1]})) \to \mathcal{G}(\mathcal{C},\mathcal{Q})$$

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Since $\mathcal{G}$ is additive, it will suffice to verify that

$$\mathsf{Q}^n(\mathcal{C},\mathfrak{P}^{[-1]}) \to \mathsf{Null}^n(\mathcal{C},\mathfrak{P}^{[-1]}) \to (\mathcal{C},\mathfrak{P})$$

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is split Poincaré-Verdier sequence. This can be deduced from the dual statement by applying to general principles, but can also be verified by hand once all definitions are unwind.