New perspectives in hermitian K-theory III

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Joint work with Baptiste Calmès, Emanuele Dotto, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus and Wolfgang Steimle.

Recollection

- A Poincaré ∞-category is a stable ∞-category $\mathcal{C}$ equipped with a quadratic functor $\Omega\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p$ satisfying a suitable non-degeneracy condition, which results in a duality $\mathrm{D}_{\Omega}\colon \mathcal{C}^{\mathrm{op}} \xrightarrow{\simeq} \mathcal{C}$.
- For a ring R and an invertible module with involution M we have a sequence of Poincaré structures on the stable ∞-category $\mathcal{D}^{\mathrm{p}}(R)$

$$\Omega^{\mathrm{q}}_M \Rightarrow \Omega^{\mathrm{gq}}_M \Rightarrow \Omega^{\mathrm{ge}}_M \Rightarrow \Omega^{\mathrm{gs}}_M \Rightarrow \Omega^{\mathrm{s}}_M$$

- For a Poincaré ∞-category we can define its Grothendieck-Witt spectrum GW and its Grothendieck-Witt space $\mathcal{GW} = \Omega^\infty \mathrm{GW}$ by means of cobordism ∞-categories.
- The functors GW and $\mathcal{GW}$ are both additive - they send split Poincaré-Verdier squares to pullback squares.
- The Grothendieck-Witt space is group-like, and so we can consider it as taking values in E_∞-groups.
- $\mathcal{GW}$ and GW are the initial functors with these properties receiving a map from Pn and $\Sigma^\infty \mathrm{Pn}$, respectively.

The L-theory space

Recall from Markus' talks the definition of the L-theory space.

Definition

For $(\mathcal{C}, \Omega)$ Poincaré, let $\rho_n(\mathcal{C}, \Omega) = (\mathrm{Fun}(\mathcal{T}_n^{\mathrm{op}}, \mathcal{C}), \Omega_{[n]})$, where

- $\mathcal{T}_n$ is the poset of non-empty subsets of $[n]$.
- $\Omega_{[n]}$ is the functor which sends $\varphi\colon \mathcal{T}_n^{\mathrm{op}} \to \mathcal{C}$ to $\lim_{\mathcal{T}_n} \Omega \circ \varphi$.

The Poincaré ∞-categories $\rho_n(\mathcal{C}, \Omega)$ fit into a simplicial object $\rho_\bullet(\mathcal{C}, \Omega)$. For a functor $\mathcal{F}\colon \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathcal{E}$ with $\mathcal{E}$ admitting geometric realizations define

$$\rho\mathcal{F}\colon \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathcal{E} \qquad \rho\mathcal{F}(\mathcal{C}, \Omega) = |\mathcal{F}\rho_\bullet(\mathcal{C}, \Omega)|.$$

Definition (Lurie-Ranicki)

The L-space of $(\mathcal{C}, \Omega)$ is define by $\mathcal{L} := \rho\mathrm{Pn}$.

- $\mathcal{L}$ is a group-like Verdier-localizing functor from $\mathrm{Cat}_\infty^{\mathrm{p}}$ to spaces.
- For $n \geq 0$ we have a natural isomorphism $\pi_n \mathcal{L}(\mathcal{C}, \Omega) \cong \mathrm{L}_n(\mathcal{C}, \Omega)$ (see Markus's talks).

From $\mathcal{GW}$ to $\mathcal{L}$

By the universal property of $\mathcal{GW}$, the map $\mathrm{Pn} \to \mathcal{L}$ factors through an essentially unique map $\mathcal{GW} \to \mathcal{L}$. This map can also be constructed explicitly as follows:

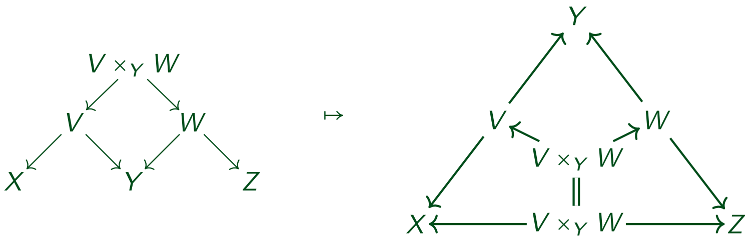
For $n \in \Delta$ consider the natural map of posets

$$\eta_n \colon \mathcal{T}_n^{\mathrm{op}} \to \mathrm{TwAr}[n] \qquad [n] \supseteq T \mapsto \min(T) \leq \max(T).$$

$\Rightarrow$ restriction yields a hermitian (and in fact Poincaré) functor

$$\mathrm{Q}_n(\mathcal{C}, \Omega) \to \rho_n(\mathcal{C}, \Omega).$$

This map is an equivalence for $n = 0, 1$. For $n = 2$ it sends



From $\mathcal{GW}$ to $\mathcal{L}$

Obtain a map

$$|\mathrm{Cob}(\mathcal{C}, \Omega)| = |\mathrm{Pn}\,\mathsf{Q}_\bullet(\mathcal{C}, \Omega^{[1]})| \to |\mathrm{Pn}\rho_\bullet(\mathcal{C}, \Omega^{[1]})| = \mathcal{L}(\mathcal{C}, \Omega^{[1]})$$

and hence a map

$$\mathcal{GW}(\mathcal{C}, \Omega) \to \Omega\,\mathcal{L}(\mathcal{C}, \Omega^{[1]}) \xrightarrow{\partial} \mathcal{L}(\mathcal{C}, \Omega)$$

where ∂ is the boundary map coming from the metabolic sequence.

Bordism invariance

A fundamental property of $\mathcal{L}$ is that it is *bordism invariant*. What does that mean?

Internal functor categories

For $(\mathcal{C}, Ϙ), (\mathcal{D}, \Phi)$ two hermitian ∞-categories define

$$\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}, Ϙ), (\mathcal{D}, \Phi)) := (\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}), \mathrm{nat}^{\Phi}_{Ϙ})$$

where $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ is the ∞-category of exact functors $\mathcal{C} \to \mathcal{D}$, and $\mathrm{nat}^{\Phi}_{Ϙ}(f) = \mathrm{nat}(Ϙ, f^*\Phi)$.

- Hermitian objects in $\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}, Ϙ), (\mathcal{D}, \Phi))$ correspond to hermitian functors $(f, \eta)\colon (\mathcal{C}, Ϙ) \to (\mathcal{D}, \Phi)$.
- If $(\mathcal{C}, Ϙ)$ and $(\mathcal{D}, \Phi)$ are Poincaré then $\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C}, Ϙ), (\mathcal{D}, \Phi))$ is Poincaré and its Poincaré objects correspond to Poincaré functors from $(\mathcal{C}, Ϙ)$ to $(\mathcal{D}, \Phi)$.

Bordism equivalences

A fundamental property of $\mathcal{L}$ is that it is *bordism invariant*. What does that mean?

Definition

- Two Poincaré functors $(f,\eta),(g,\vartheta)\colon(\mathcal{C},\Omega)\to(\mathcal{D},\Phi)$ are said to be *cobordant* if there is a cobordism between them when considered as Poincaré objects in $\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\Omega),(\mathcal{D},\Phi))$.
- A Poincaré functor $(f,\eta)\colon(\mathcal{C},\Omega)\to(\mathcal{D},\Phi)$ is called a *bordism equivalence* there exists a Poincaré functor $(g,\theta)\colon(\mathcal{D},\Phi)\to(\mathcal{C},\Omega)$ such that $(f,\eta)\circ(g,\theta)$ and $(g,\theta)\circ(f,\eta)$ are cobordant to the respective identifies.

Example

If $(\mathcal{C},\Omega)$ is a Poincaré ∞-category and $\mathcal{A}\subseteq\mathcal{C}$ is an isotropic subcategory then the Poincaré functor $\mathrm{Hlgy}(\mathcal{A})\to(\mathcal{C},\Omega)$ is a bordism equivalence.

Bordism invariant functors

Definition

A functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ is called *bordism invariant* if it sends bordism equivalences to equivalences.

For group-like additive functors bordism invariance admits several equivalent characterizations:

Lemma

For $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{E}$ a group-like additive functor the following are equivalent:

1. *$\mathcal{F}$ is bordism invariant.*
2. *$\mathcal{F}$ vanishes on metabolic Poincaré ∞-categories, i.e., those that admit a Lagrangian.*
3. *$\mathcal{F}$ vanishes on $\mathrm{Hyp}(\mathcal{C})$ for every $\mathcal{C} \in \mathrm{Cat}^{\mathrm{ex}}_{\infty}$.*

Proof.

(1) implies (2) implies (3) without the assumptions on $\mathcal{F}$. To prove (3) $\Rightarrow$ (1) one uses the isotropic decomposition theorem. □

Bordism invariance of $\mathcal{L}$

$\mathcal{L}$ is group-like and Verdier-localizing. To see that it is bordism invariant it hence suffices to check that it vanishes on hyperbolic Poincaré ∞-categories.

The simplicial space

$$\mathrm{Pn}(\rho_\bullet \operatorname{Hyp}(\mathcal{C})) \simeq \mathrm{Pn}\operatorname{Hyp}(\operatorname{Fun}(\mathcal{T}_\bullet^{\mathrm{op}}, \mathcal{C})) \simeq \operatorname{Fun}(\mathcal{T}_\bullet^{\mathrm{op}}, \mathcal{C})^{\simeq}$$

is the simplicial subdivision of $\operatorname{Fun}(\Delta^\bullet, \mathcal{C})^{\simeq}$, and hence

$$\mathcal{L}(\operatorname{Hyp}(\mathcal{C})) \simeq |\operatorname{Fun}(\mathcal{T}_\bullet^{\mathrm{op}}, \mathcal{C})^{\simeq}| \simeq |\operatorname{Fun}(\Delta^\bullet, \mathcal{C})^{\simeq}| \simeq |\mathcal{C}| \simeq *$$

since $\mathcal{C}$ has a zero object.

An L-theory spectrum

Recall from the previous talk the operation $\mathcal{F} \mapsto \mathbb{C}\mathrm{ob}^{\mathcal{F}}$ which sends a group-like additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$ to an additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ such that $\Omega^{\infty}\mathbb{C}\mathrm{ob}^{\mathcal{F}} \simeq \mathcal{F}$.

Definition

We define the L-spectrum of a Poincaré ∞-category by

$$\mathrm{L}(\mathcal{C}, \Qoppa) = \mathbb{C}\mathrm{ob}^{\mathcal{L}}(\mathcal{C}, \Qoppa)$$

- L is by construction an additive functor to spectra (automatically group-like since $\mathcal{S}p$ is stable).
- The Q-construction preserves hyperbolic Poincaré ∞-categories $\Rightarrow$ the functor L is bordism invariant.
- The map $\mathcal{GW} \to \mathcal{L}$ induces a map $\mathrm{GW} \to \mathrm{L}$.

Shifts of L-theory

Applying L to the metabolic sequence yields a fiber sequence

$$\mathrm{L}(\mathcal{C}, \Qoppa^{[-1]}) \to \mathrm{L}(\mathrm{Met}(\mathcal{C}, \Qoppa)) \to \mathrm{L}(\mathcal{C}, \Qoppa).$$

But $\mathrm{L}(\mathrm{Met}(\mathcal{C}, \Qoppa)) \simeq 0$, so $\mathrm{L}(\mathcal{C}, \Qoppa^{[-1]}) \simeq \Omega\,\mathrm{L}(\mathcal{C}, \Qoppa)$. By induction:

$$\mathrm{L}(\mathcal{C}, \Qoppa^{[-n]}) \simeq \Omega^n\,\mathrm{L}(\mathcal{C}, \Qoppa) \qquad n \in \mathbb{Z}.$$

Conclusion

$\mathrm{L}(\mathcal{C}, \Qoppa)$ can also be described as the Ω-spectrum with components

$$\mathcal{L}(\mathcal{C}, \Qoppa), \mathcal{L}(\mathcal{C}, \Qoppa^{[1]}), \mathcal{L}(\mathcal{C}, \Qoppa^{[2]}), \ldots,$$

with the structure maps $\mathcal{L}(\mathcal{C}, \Qoppa^{[n]}) \xrightarrow{\simeq} \Omega\,\mathcal{L}(\mathcal{C}, \Qoppa^{[n+1]})$ coming from the metabolic sequence.

This observation holds for any group-like additive functor $\mathcal{F}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}$. In fact, bordism invariant group-like additive functors to spaces admit an essentially unique bordism invariant additive lift to spectra, given by the above formula.

Bordification of additive functors

Definition

Let $b\colon \mathcal{F} \to \mathcal{F}'$ be a map between additive functors $\mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$. We say that b exhibits $\mathcal{F}'$ as the *bordification* of $\mathcal{F}$ if $\mathcal{F}'$ is bordism invariant and for every other bordism invariant additive functor $\mathcal{G}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ the induced map

$$\mathsf{nat}(\mathcal{F}', \mathcal{G}) \to \mathsf{nat}(\mathcal{F}, \mathcal{G})$$

is an equivalence.

Define $\mathsf{GW}^{\mathrm{bord}}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$ to be the cofiber of the map

$$\mathsf{K}_{\mathrm{hC}_2} \to \mathsf{GW}$$

induced from the C_2-equivariant hyperbolic map $\mathrm{hyp}\colon \mathsf{K} \to \mathsf{GW}$. Note that $\mathsf{GW}^{\mathrm{bord}}$ is additive since $\mathcal{S}p$ is stable.

Lemma

The map $\mathsf{GW} \to \mathsf{GW}^{\mathrm{bord}}$ exhibits $\mathsf{GW}^{\mathrm{bord}}$ as the bordification of GW.

Lemma

The map $\mathsf{GW} \to \mathsf{GW}^{\mathrm{bord}}$ *exhibits* $\mathsf{GW}^{\mathrm{bord}}$ *as the bordification of* GW*.*

Proof.

We first check that $\mathsf{GW}^{\mathrm{bord}}$ is bordism invariant. Since it is an additive functor to spectra it will suffice to check that it vanishes on $\mathsf{Hyp}(\mathcal{C})$ for every $\mathcal{C}$. Indeed, this is because the map

$$\mathsf{K}(\mathcal{C} \times \mathcal{C}^{\mathrm{op}})_{\mathrm{hC_2}} = [\mathsf{K}(\mathcal{C}) \times \mathsf{K}(\mathcal{C})]_{\mathrm{hC_2}} \to \mathsf{K}(\mathcal{C}) = \mathsf{GW}(\mathsf{Hyp}(\mathcal{C}))$$

is an equivalence. We now check that $\mathsf{GW} \to \mathsf{GW}^{\mathrm{bord}}$ induces an equivalence on mapping spectra into a bordism invariant additive test functor $\mathcal{G}$. For this it suffices to show that $\mathsf{K}_{\mathrm{hC_2}}$ maps trivially to any bordism invariant functor, or equivalently, that K has that property.

- Fact: the functor $(\mathcal{C}, \mathfrak{Q}) \mapsto \mathsf{Hyp}(\mathcal{C})$ from $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ to $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ is adjoint to itself in both directions

$\Rightarrow$ maps from $\mathsf{K} = \mathsf{GW} \circ \mathsf{Hyp}$ to $\mathcal{G}$ are like maps from GW to $\mathcal{G} \circ \mathsf{Hyp} = 0$. □

L-theory as bordification

We claim that $\mathsf{GW}^{\mathrm{bord}}$ is actually equivalent to L.

Two birds with one stone

This gives a universal characterization for L, and at the same time puts L in an exact sequence with GW and $\mathsf{K}_{\mathrm{hC}_2}$.

We proceed in several steps:

- L is naturally equivalent to $\rho\,\mathsf{GW} = |\,\mathsf{GW}(\rho_\bullet(-))|$, such that the map $\mathsf{GW} \to \mathsf{L}$ identifies with the one induced by the inclusion of zero simplices. This is proven using the commutativity $\rho_n\,\mathsf{Q}_m \simeq \mathsf{Q}_m\,\rho_n$.
- For $\mathcal{G}$ an additive bordism invariant functor the map $\mathcal{G} \to \rho\mathcal{G}$ is an equivalence. This follows from the fact that each simplicial structure map $\rho_n(\mathcal{C}, \Omega) \to \rho_m(\mathcal{C}, \Omega)$ is a bordism equivalence.
- The functor $\rho\,\mathsf{K}$ is trivial. Indeed, by the commutativity of the ρ and hyperbolic constructions, $\rho\,\mathsf{K}(\mathcal{C}) \simeq \rho\,\mathsf{GW}(\mathrm{Hyp}(\mathcal{C})) \simeq \mathsf{L}(\mathrm{Hyp}(\mathcal{C})) \simeq 0$.

Proposition

There is a natural equivalence $\mathrm{GW}^{\mathrm{bord}} \simeq \mathrm{L}$.

Proof.

Apply ρ to the sequence

$$\mathrm{K}_{\mathrm{hC}_2} \to \mathrm{GW} \to \mathrm{GW}^{\mathrm{bord}}$$

to obtain a diagram

$$\begin{array}{ccccccc}
 & & \mathrm{K}_{\mathrm{hC}_2} & \longrightarrow & \mathrm{GW} & \longrightarrow & \mathrm{GW}^{\mathrm{bord}} \\
 & & \downarrow & & \downarrow & & \downarrow \simeq \\
0 \simeq (\rho\,\mathrm{K})_{\mathrm{hC}_2} & \xrightarrow{\simeq} & \rho(\mathrm{K}_{\mathrm{hC}_2}) & \longrightarrow & \mathrm{L} & \xrightarrow{\simeq} & \rho\,\mathrm{GW}^{\mathrm{bord}}
\end{array}$$

where we identified $\rho\,\mathrm{GW} \simeq \mathrm{L}$ by the first point in the previous slide. □

The universal property of L-theory

Corollary

The natural transformation $\mathsf{GW} \Rightarrow \mathsf{L}$ *exhibits* L *as the initial bordism invariant additive functor under* GW.

Corollary (Universality for L-theory)

The natural transformation $\Sigma^\infty \mathrm{Pn} \Rightarrow \mathsf{L}$ *exhibits* L *as the initial bordism invariant additive functor under* $\Sigma^\infty \mathrm{Pn}$.

Long exact sequences of GW and L-groups

Corollary (The fundamental fiber sequence)

The natural maps

$$\mathsf{K}_{\mathrm{hC}_2} \to \mathsf{GW} \to \mathsf{L}$$

form a fiber sequence of additive functors $\mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathcal{S}p$.

- Since K and L are Verdier-localizing - it follows that GW is Verdier localizing as well.
- Gives a long exact sequence in homotopy groups

$$\cdots \to \pi_n \mathsf{K}(\mathcal{C})_{\mathrm{hC}_2} \to \mathsf{GW}_n(\mathcal{C}, \Omega) \to \mathsf{L}_n(\mathcal{C}, \Omega) \to \pi_{n-1} \mathsf{K}(\mathcal{C})_{\mathrm{hC}_2} \to \mathsf{GW}_{n-1}(\mathcal{C}, \Omega) \to \mathsf{L}_{n-1}(\mathcal{C}, \Omega) \to \cdots$$

The Tate square

$$\begin{array}{ccc} \mathsf{GW} & \longrightarrow & \mathsf{K}^{\mathrm{hC}_2} \\ \downarrow & & \downarrow \\ \mathsf{L} & \longrightarrow & \mathsf{K}^{\mathrm{tC}_2} \simeq \rho(\mathsf{K}^{\mathrm{hC}_2}) \end{array}$$

Applications

Extending the classical exact sequence

For R a ring and M an invertible module with involution over R, recall from the first talk the exact sequence

$$\mathrm{K}_0(R)_{\mathrm{C}_2} \to \mathrm{GW}_0^{\mathrm{q}}(R, M) \to \mathrm{L}_0^{\mathrm{q}}(R, M) = \mathrm{W}_0^{\mathrm{q}}(R, M)$$

When 2 is invertible in R

- The quadratic, symmetric, and all the genuine structures we described on $\mathcal{D}^{\mathrm{p}}(R)$ are all the same, call them Ω_M.
- In this case $\mathrm{GW}(\mathcal{D}^{\mathrm{p}}(R), \Omega_M)$ coincides with Schlichting's GW-spectrum, and $\mathrm{L}(\mathcal{D}^{\mathrm{p}}(R), \Omega_M)$ coincides with Ranicki's quadratic L-spectrum.
- The above sequence extends to a long exact sequence involving GW-groups and L-groups as in the previous slide.
- This was known in this case by the work of Schlichting.

Extending the classical exact sequence

For R a ring and M an invertible module with involution over R, recall from the first talk the exact sequence

$$\mathrm{K}_0(R)_{\mathrm{C}_2} \to \mathrm{GW}_0^{\mathrm{q}}(R, M) \to \mathrm{L}_0^{\mathrm{q}}(R, M) = \mathrm{W}_0^{\mathrm{q}}(R, M)$$

When 2 is not invertible in R

- The classical groups $\mathrm{GW}_0^{\mathrm{q}}(R, M)$ and $\mathrm{L}_0^{\mathrm{q}}(R, M)$ are the 0-th Grothendieck-Witt and L-group of both $\Qoppa_M^{\mathrm{q}}$ and $\Qoppa_M^{\mathrm{gq}}$, which agree in low degrees.
- Taking either $\Qoppa_M^{\mathrm{q}}$ or $\Qoppa_M^{\mathrm{gq}}$, we may extend this sequence in two ways.
- The first option involves the classical quadratic L-groups, but not the classical quadratic Grothendieck-Witt groups.
- The second option involves the classical quadratic Grothendieck-Witt groups, but not the classical quadratic L-groups.
- This mismatch explains why this sequence remained non-extendable from a classical perspective.

Genuine L-groups

Question

Taking Ω^{gq}_M to get classical GW-groups, what are the L-groups we get?

We have equivalences of Poincaré ∞-categories

$$(\mathcal{D}^p(R), \Omega^{gq}_M) \simeq (\mathcal{D}^p(R), (\Omega^{ge}_{-M})^{[2]}) \simeq (\mathcal{D}^p(R), (\Omega^{gs}_M)^{[4]}).$$

By bordism invariance

$$\mathrm{L}^{gq}(R, M) \simeq \Sigma^2 \mathrm{L}^{ge}(R, -M) \simeq \Sigma^4 \mathrm{L}^{gs}(R, M)$$

$$\Rightarrow$$

$$\mathrm{L}^{gq}_n(R, M) \simeq \mathrm{L}^{ge}_{n-2}(R, -M) \simeq \mathrm{L}^{gs}_{n-4}(R, M)$$

Conclusion

To describe the L-groups of all three genuine structures $\Omega^{gq}_M, \Omega^{ge}_M, \Omega^{gs}_M$, it suffices to describe the genuine symmetric L-groups for $\pm M$.

Genuine L-groups

Summary

To describe the L-groups of all three genuine structures $\Upsilon^{\mathrm{gq}}_M, \Upsilon^{\mathrm{ge}}_M, \Upsilon^{\mathrm{gs}}_M$, it suffices to describe the genuine symmetric L-groups for $\pm M$.

Proposition

- *For $n \geq 0$ the groups $\mathrm{L}^{\mathrm{gs}}_n(R, M)$ are naturally isomorphic to Ranicki's (M-based) symmetric L-groups of short complexes.*
- $\mathrm{L}^{\mathrm{gs}}_{-2}(R, M)$ *and* $\mathrm{L}^{\mathrm{gs}}_{-1}(R, M)$ *can be described as Witt groups of even forms and even formations, respectively.*
- *For $n \leq -3$ the groups $\mathrm{L}^{\mathrm{gs}}_n(R, M)$ are naturally isomorphic to the classical quadratic L-groups.*

Remark

Somewhat strikingly, Ranicki himself extended his short L-groups in negative degrees by defining them exactly as above. The genuine L-theory perspective then unites these various ad hoc definitions, and shows that they fit together as the homotopy groups of a spectrum.

Rings with bounded global dimension

Recall that R is said to have global dimension $\leq d$ if every module admits a projective resolution of length $\leq d$.

Theorem (L-theory comparison beyond the global dimension)

Let R be a Noetherian ring of global dimension $\leq d$. Then the maps

$$\mathrm{L}_n^{\mathrm{gq}}(R, M) \to \mathrm{L}_n^{\mathrm{ge}}(R, M) \to \mathrm{L}_n^{\mathrm{gs}}(R, M) \to \mathrm{L}_n^{\mathrm{s}}(R, M)$$

are isomorphisms for n at least $d+3, d+1$ and $d-1$, respectively (and injective for n at least $d+2, d$ and $d-2$, respectively).

Corollary (GW-comparison beyond the global dimension)

Let R be a Noetherian ring of global dimension $\leq d$. Then the maps

$$\mathrm{GW}_n^{\mathrm{gq}}(R, M) \to \mathrm{GW}_n^{\mathrm{ge}}(R, M) \to \mathrm{GW}_n^{\mathrm{gs}}(R, M) \to \mathrm{GW}_n^{\mathrm{s}}(R, M)$$

are isomorphisms for n at least $d+3, d+1$ and $d-1$, respectively. The first three groups are classical!

Devissage for homotopy symmetric forms

Let R be a Dedekind domain with fraction field K, S a set of prime ideals. Define $R_S \subseteq K$ to be the localization of R away from S. Let M be a line bundle with an R-linear involution.

Claim

$$(\mathcal{D}^{\mathrm{p}}(R)_S, \iota^* \Upsilon^{\mathrm{s}}_M) \to (\mathcal{D}^{\mathrm{p}}(R), \Upsilon^{\mathrm{s}}_M) \to (\mathcal{D}^{\mathrm{p}}(R_S), \Upsilon^{\mathrm{s}}_{M_S})$$

is a Poincaré-Verdier sequence. Here $\iota\colon \mathcal{D}^{\mathrm{p}}(R)_S \subseteq \mathcal{D}^{\mathrm{p}}(R)$ consists of S-torsion complexes and the Poincaré structure on it restricts from Υ^{s}_M.

Proposition (Devissage)

There is a natural Poincaré functor

$$\bigoplus_{p \in S} (\mathcal{D}^{\mathrm{p}}(\mathbb{F}_p), (\Upsilon^{\mathrm{s}}_{M/p})^{[-1]}) \to (\mathcal{D}^{\mathrm{p}}(R)_S, \iota^* \Upsilon^{\mathrm{s}}_M)$$

which induces an equivalences on K, GW *and* L.

Localization sequences for Dedekind rings

Let R be a Dedekind domain, S a set of prime ideals. Define R_S to be the localization of R away from S. Let M be a line bundle with an R-linear involution.

Corollary

There are exact sequences

$$\bigoplus_{p \in S} \Omega \, \mathrm{L}^{\mathrm{s}}(\mathbb{F}_p, M/p) \to \mathrm{L}^{\mathrm{s}}(R, M) \to \mathrm{L}^{\mathrm{s}}(R_S, M_S)$$

and

$$\bigoplus_{p \in S} \mathrm{GW}^{\mathrm{s}}(\mathbb{F}_p, M/p[-1]) \to \mathrm{GW}^{\mathrm{s}}(R, M) \to \mathrm{GW}^{\mathrm{s}}(R_S, M_S)$$

These sequences induces long exact sequences in homotopy symmetric Grothendieck-Witt and L-groups.The latter coincide with the genuine symmetric variants in non-negative degrees since Dedekind rings have global dimension 1.

The homotopy limit problem for number rings

Question (Thomason)

When is the map $\mathcal{GW}^{s}_{cl}(R, M) \to \mathcal{K}(R, M)^{hC_2}$ *a* 2*-adic equivalence?*

State of the art

- True for finite fields (Friedlander, Fiedorowicz-Priddy).
- True for R a field of characteristic 0 with $\mathrm{vcd}_2 < 0$ (Hu-Kriz-Ormsby, another proof was recently given by Bachmann and Hopkin).
- True for R a commutative $\mathbb{Z}[1/2]$-algebra with a global bound on vcd_2 of its residue fields (Berrick-Karoubi-Schlichting-Østvær).

Theorem (The homotopy limit problem for number rings)

Let R *be a Dedekind domain whose fraction field is a number field,* M *a line bundle over* R *with an* R*-linear involution. Then the map*

$$\mathrm{GW}^{s}(R, M) \to \mathrm{K}(R, M)^{hC_2}$$

is a 2*-adic equivalence.*

The Grothendieck-Witt groups of the integers

What are the higher Grothendieck-Witt groups of the integers?

Berrick-Karoubi

A computation of the ± 1 Grothendieck-Witt groups of $\mathbb{Z}[\frac{1}{2}]$.

- By the localization sequence and using the fact that $\mathrm{GW}^{\mathrm{s}}(\mathbb{F}_2)$ has only odd torsion in positive degrees, the difference between $\mathrm{GW}^{\mathrm{s}}(\mathbb{Z})$ and $\mathrm{GW}^{\mathrm{s}}(\mathbb{Z}[\frac{1}{2}])$ consists of odd torsion.
- Since the symmetric L-groups of $\mathbb{Z}$ contain no odd torsion, the odd torsion of $\mathrm{GW}^{\mathrm{s}}_n(\mathbb{Z})$ is the same as the odd torsion of $\pi_n \mathrm{K}(\mathbb{Z})_{\mathrm{hC}_2}$.
- This requires knowing the C_2-action on $\mathrm{K}_n(\mathbb{Z})$ (this action is the same for $Ϙ^{\mathrm{s}}_{\mathbb{Z}}$ and $Ϙ^{\mathrm{s}}_{-\mathbb{Z}}$).

Proposition

For $n \geq 2$ the C_2-action on $\mathrm{K}_{2n-1}(\mathbb{Z})[\frac{1}{2}]$ and $\mathrm{K}_{2n-2}(\mathbb{Z})[\frac{1}{2}]$ is given by multiplication by $(-1)^n$.

Let B_n be the n'th Bernoulli number. Write c_n and w_{2n} for the numerator and denominator of $\left|\frac{B_{2n}}{4n}\right|$, respectively.

Theorem

The classical ε-symmetric Grothendieck-Witt groups $\mathbb{Z}$ are given in degrees $n \geq 1$ by the following table:

$n =$	$\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl},n}(\mathbb{Z})$	$\mathrm{GW}^{-\mathrm{s}}_{\mathrm{cl},n}(\mathbb{Z})$
$8k$	$\mathbb{Z} \oplus \mathbb{Z}/2$	0
$8k+1$	$(\mathbb{Z}/2)^3$	0
$8k+2$	$(\mathbb{Z}/2)^2 \oplus \mathrm{K}_{8k+2}(\mathbb{Z})_{\mathrm{odd}}$	$\mathbb{Z} \oplus \mathrm{K}_{8k+2}(\mathbb{Z})_{\mathrm{odd}}$
$8k+3$	$\mathbb{Z}/w_{4k+2}$	$\mathbb{Z}/2w_{4k+2}$
$8k+4$	$\mathbb{Z}$	$\mathbb{Z}/2$
$8k+5$	0	$\mathbb{Z}/2$
$8k+6$	$\mathrm{K}_{8k+6}(\mathbb{Z})_{\mathrm{odd}}$	$\mathbb{Z} \oplus \mathrm{K}_{8k+6}(\mathbb{Z})_{\mathrm{odd}}$
$8k+7$	$\mathbb{Z}/w_{4k+4}$	$\mathbb{Z}/w_{4k+4}$

The group $\mathrm{K}_{4m-2}(\mathbb{Z})_{\mathrm{odd}}$ has order c_m and is known to be cyclic for $m \leq 5000$ (Weibel). This holds for all m if Vandiver's conjecture is true.

Example

Table: the first 24 symmetric Grothendieck-Witt groups of $\mathbb{Z}$

k	$\mathrm{GW}_k^s(\mathbb{Z})$	k	$\mathrm{GW}_k^s(\mathbb{Z})$	k	$\mathrm{GW}_k^s(\mathbb{Z})$
0	$\mathbb{Z} \oplus \mathbb{Z}$	8	$\mathbb{Z} \oplus \mathbb{Z}/2$	16	$\mathbb{Z} \oplus \mathbb{Z}/2$
1	$(\mathbb{Z}/2)^3$	9	$(\mathbb{Z}/2)^3$	17	$(\mathbb{Z}/2)^3$
2	$(\mathbb{Z}/2)^2$	10	$(\mathbb{Z}/2)^2$	18	$(\mathbb{Z}/2)^2$
3	$\mathbb{Z}/24$	11	$\mathbb{Z}/504$	19	$\mathbb{Z}/264$
4	$\mathbb{Z}$	12	$\mathbb{Z}$	20	$\mathbb{Z}$
5	0	13	0	21	0
6	0	14	0	22	$\mathbb{Z}/691$
7	$\mathbb{Z}/240$	15	$\mathbb{Z}/480$	23	$\mathbb{Z}/65520$

Example

Table: the first 24 skew-symmetric Grothendieck-Witt groups of $\mathbb{Z}$

k	$GW_k^{-s}(\mathbb{Z})$	k	$GW_k^{-s}(\mathbb{Z})$	k	$GW_k^{-s}(\mathbb{Z})$
0	$\mathbb{Z}$	8	0	16	0
1	0	9	0	17	0
2	$\mathbb{Z}$	10	$\mathbb{Z}$	18	$\mathbb{Z}$
3	$\mathbb{Z}/48$	11	$\mathbb{Z}/1008$	19	$\mathbb{Z}/528$
4	$\mathbb{Z}/2$	12	$\mathbb{Z}/2$	20	$\mathbb{Z}/2$
5	$\mathbb{Z}/2$	13	$\mathbb{Z}/2$	21	$\mathbb{Z}/2$
6	$\mathbb{Z}$	14	$\mathbb{Z}$	22	$\mathbb{Z} \oplus \mathbb{Z}/691$
7	$\mathbb{Z}/240$	15	$\mathbb{Z}/480$	23	$\mathbb{Z}/65520$

The quadratic Grothendieck-Witt groups of the integers

Theorem

The classical quadratic Grothendieck-Witt groups of the integers are given by

- $\mathrm{GW}_0^{\mathrm{gq}}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$.
- $\mathrm{GW}_1^{\mathrm{gq}}(\mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.
- $\mathrm{GW}_n^{\mathrm{gq}}(\mathbb{Z}) = \mathrm{GW}_n^{\mathrm{gs}}(\mathbb{Z})$ *for* $n \geq 2$.

Theorem

The classical skew-quadratic Grothendieck-Witt groups of the integers are given by

- $\mathrm{GW}_0^{-\mathrm{gq}}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$.
- $\mathrm{GW}_1^{-\mathrm{gq}}(\mathbb{Z}) = \mathbb{Z}/4$.
- $\mathrm{GW}_2^{-\mathrm{gq}}(\mathbb{Z}) = \mathbb{Z}$.
- $\mathrm{GW}_3^{-\mathrm{gq}}(\mathbb{Z}) = \mathbb{Z}/24$.
- $\mathrm{GW}_n^{-\mathrm{gq}}(\mathbb{Z}) = \mathrm{GW}_n^{-\mathrm{gs}}(\mathbb{Z})$ *for* $n \geq 4$.