Universität
Münster

Metric inequalities under lower scalar curvature bounds

Mathematics Münster Mid-term Conference

Rudolf Zeidler

March 25, 2024

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Funded by the European Union
European Research Council
Established by the European Commission
Funded by DFG Deutsche Forschungsgemeinschaft
German Research Foundation
MM Mathematics Münster
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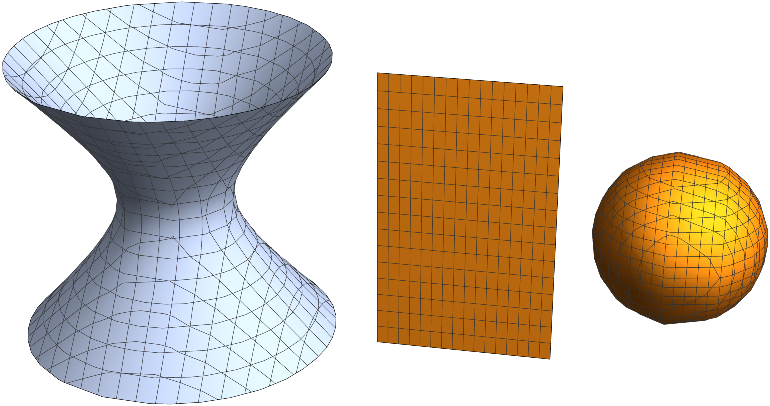

$K < 0$ $K = 0$ $K > 0$

Sectional curvature

(M, g) ... n-dimensional Riemannian manifold.

- $E \subseteq T_p M$ 2-plane $\rightsquigarrow \sec(E) \in \mathbb{R}$, defined "in terms of g, ∂g, $\partial^2 g$".

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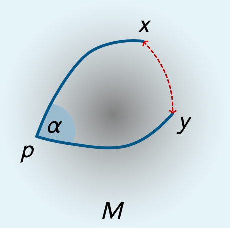
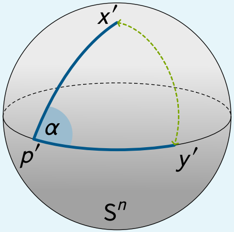
Theorem (Toponogov 1959, case of $\sec \geqslant 1$)

Let (M, g) *be complete with* $\sec \geqslant 1$. *Given "hinges"* (p, x, y) *in* M *and* (p', x', y') *in* S^n *with* $d(p, x) = d_{S^n}(p', x')$, $d(p, y) = d_{S^n}(p', y')$, $\angle(\overline{px}, \overline{py}) = \angle(\overline{p'x'}, \overline{p'y'})$, *we have* $d(x, y) \leqslant d_{S^n}(x', y')$





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Theorem (Gromoll–Meyer 1969)

Let (M, g) *complete, non-compact, with* $\sec > 0$. *Then* M *is diffeomorphic to* $\mathbb{R}^n$.

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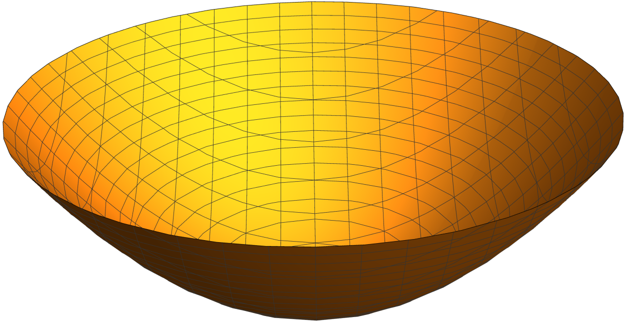

e.g. Paraboloid $\{z = x^2 + y^2\}$

Ricci curvature

- $v \in T_p M$ unit vector $\rightsquigarrow \mathrm{Ric}(v) = \sum_{i=1}^{n} \sec(\langle e_i, v \rangle)(1 - g(e_i, v)^2)$

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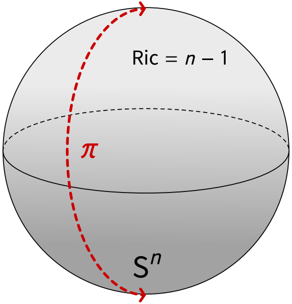
Let (M, g) *be a complete Riemannian* n*-manifold. If* $\mathrm{Ric}_g \geqslant (n-1)$, *then* $\mathrm{diam}(M, g) \leqslant \pi$.



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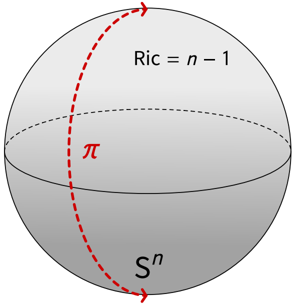
If $\mathrm{diam}(M, g) = \pi$, *then* $(M, g) \cong (S^n, g_{\mathrm{round}})$.



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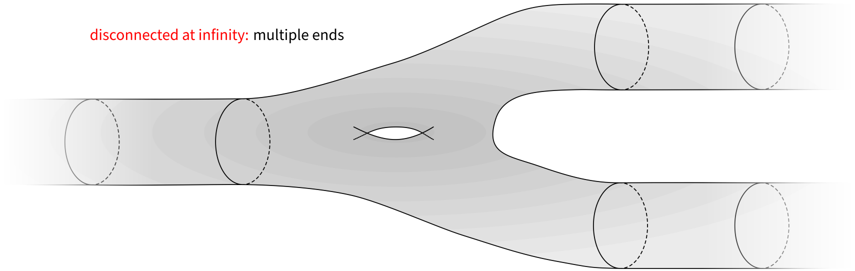

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Product constructions

$(X, g_X), (Y, g_Y) \quad \rightsquigarrow \quad$ On $X \times Y$: $\quad \mathrm{scal}_{g_X \oplus g_Y}(x, y) = \mathrm{scal}_{g_X}(x) + \mathrm{scal}_{g_Y}(y)$

- $\exists$ complete (M, g) of scal $\geqslant c > 0$ and arbitrary diameter, e.g. $S^{n-1} \times R \cdot S^1$.
- $\exists$ complete non-compact (M, g) of scal > 0 and more than one end, e.g. $S^{n-1} \times \mathbb{R}$.

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Questions

- Metric inequalities under lower scalar curvature bounds?
- Global structure of non-compact complete manifolds with scal > 0?

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K3 surface
$V^4 = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_0^4 + z_1^4 + z_3^4 + z_4^4 = 0\}$

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Minimal hypersurfaces
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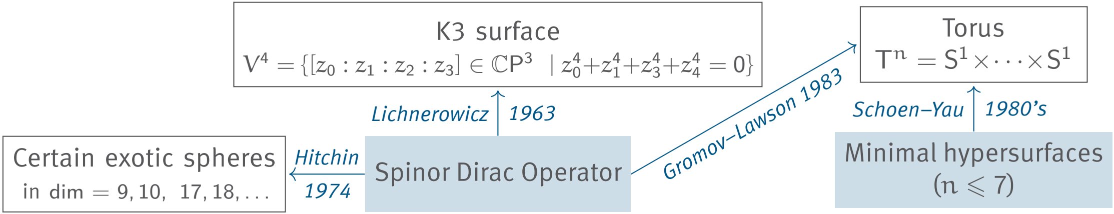

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Certain exotic spheres
in $\dim = 9, 10,\ 17, 18, \ldots$

← *Hitchin* *1974*

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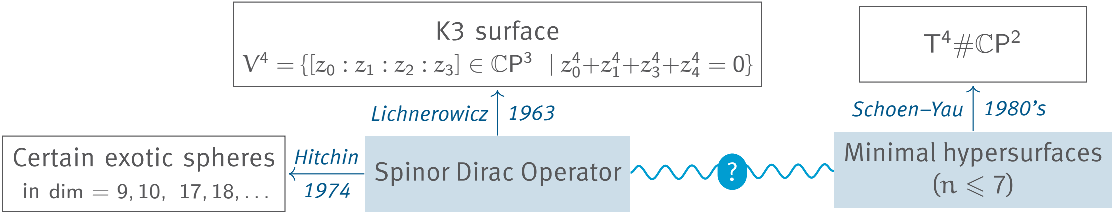

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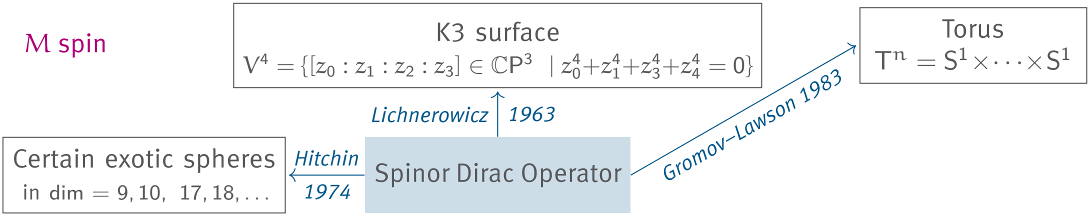

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$$\mathrm{index}(\not{D}_M) = \hat{A}(M) \neq 0 \in \mathbb{Z}$$
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$$\alpha_{\pi_1(M)}(M) \neq 0 \in KO_*(C^*(\pi_1 M))$$

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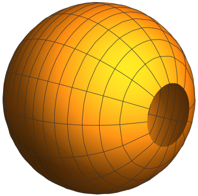

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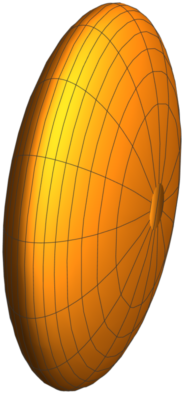

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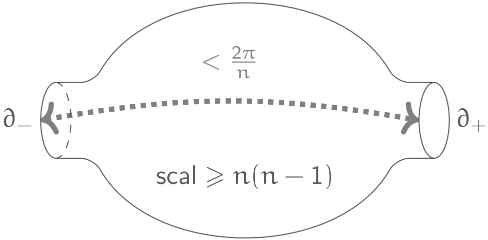
Width Conjecture (Gromov 2018)

Let M be a compact connected manifold of dimension $n-1 \neq 4$ such that M does not admit a metric of scal > 0. Let g be a Riemannian metric on $V = M \times [-1, 1]$ of $\mathrm{scal}_g \geqslant n(n-1)$. Then

$$\mathrm{width}(V, g) := \mathrm{dist}_g(\partial_- V, \partial_+ V) < \frac{2\pi}{n}, \qquad \text{where } \partial_\pm V = M \times \{\pm 1\}.$$



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- Z. 2019–2020: True if M spin and $\alpha_{\pi_1(M)}(M) \neq 0 \in KO_*(C^*(\pi_1 M))$. Thus it holds for

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Rigidity with boundary conditions

Theorem (Cecchini–Z. 2021)

Let (V, g) be a Riemannian spin manifold, $\partial V = \partial_- V \sqcup \partial_+ V$ where $\partial_\pm V$ are non-empty unions of components. Suppose $\widehat{A}(\partial_\pm V) \neq 0$ and $\mathrm{scal}_g \geqslant n(n-1)$.

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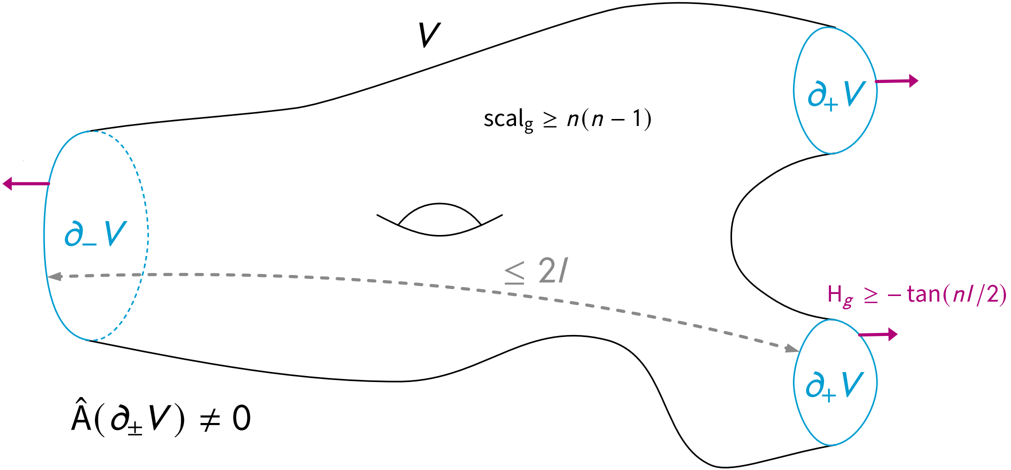

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Key proof idea: Modified Dirac operator $\mathcal{B}_f := \mathcal{D} + f\sigma$ depending on a scalar function f.

$$\rightsquigarrow \quad \mathcal{B}_f^2 = \quad \dots \quad \geqslant \widetilde{\nabla}^* \widetilde{\nabla} + \frac{\mathrm{scal}_g}{4} + \frac{n-1}{n} \left(f^2 - |df|\right).$$

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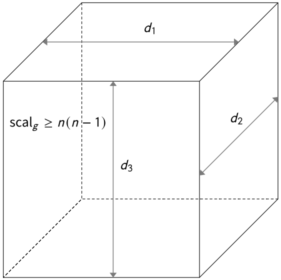
Theorem (Gromov, Wang–Xie–Yu 2021)

Let g *be a Riemannian metric on the cube* $[-1,1]^n$ *of* $\mathrm{scal}_g \geqslant n(n-1)$. *Then*

$$\sum_{i=1}^{n} \frac{1}{d_i^2} \geqslant \frac{n^2}{4\pi^2}, \qquad \text{in particular } \min_i d_i \leqslant \frac{2\pi}{\sqrt{n}}.$$



Structure of manifolds with multiple ends and $\mathrm{scal} > 0$?

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Recall: $\mathrm{Ric} > 0 \implies$ connected at infinity; but $\mathrm{scal} > 0$ allows more than one end (e.g. $S^2 \times \mathbb{R}$).

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Conjecture (Rosenberg–Stolz 1994)

Let M be a compact connected manifold that does not admit a metric of $\mathrm{scal} > 0$ with $\dim(M) \neq 4$. Then $M \times \mathbb{R}$ does not admit a complete metric of $\mathrm{scal} > 0$.

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An *open band* is a non-compact manifold V with a decomposition $\mathrm{Ends}(V) = \mathcal{E}_- \sqcup \mathcal{E}_+$, $\mathcal{E}_\pm \neq \emptyset$ open.

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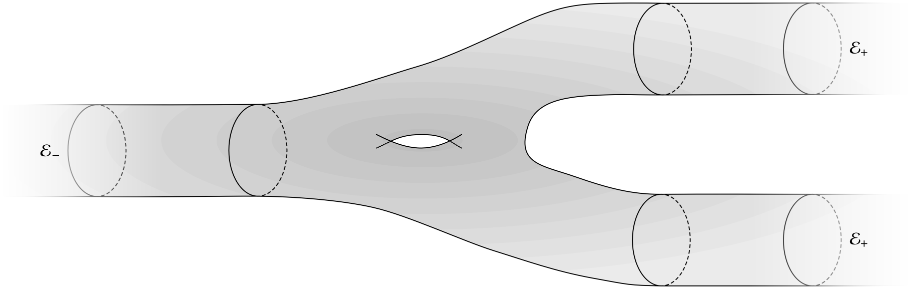

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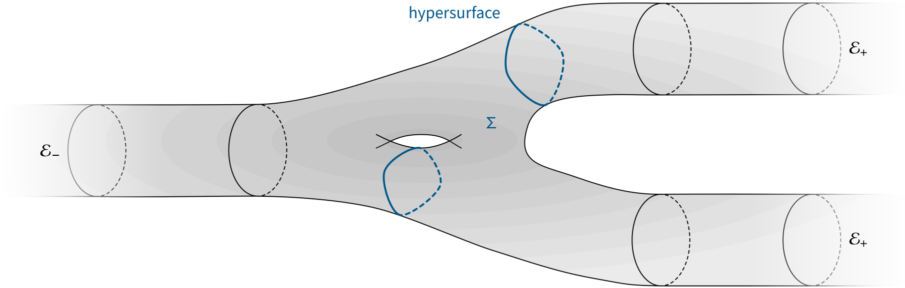

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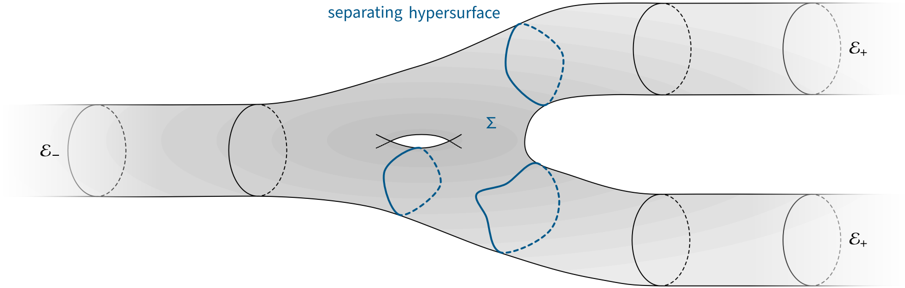

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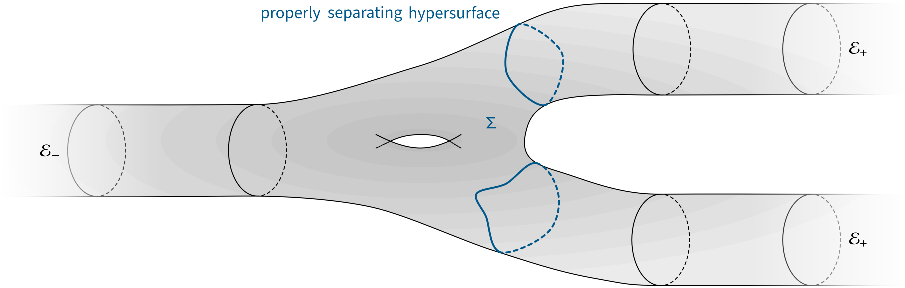

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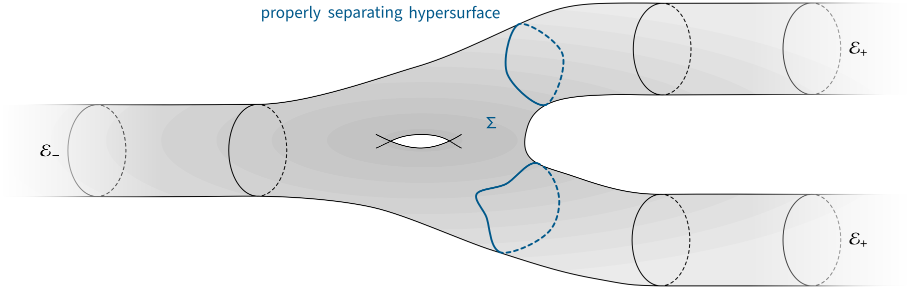

Proposition (Cecchini–Räde–Z. 2022)

Let (V, g) be an open band endowed with a complete metric of scal > 0 and $n = \dim(V) \leqslant 7$. Then there exists a properly separating $\Sigma \subset V$ that admits a metric of scal > 0.

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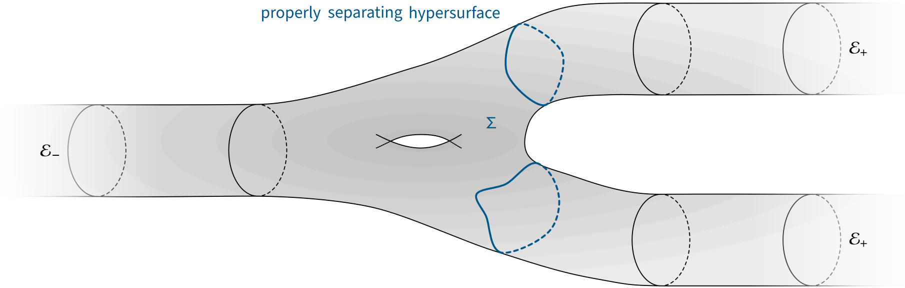

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Key proof ingredient: "μ-bubbles" (Gromov, J. Zhu, Chodosh–Li, ...; (Andersson–Eichmair–Metzger))

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*Let X be an orientable connected n-manifold with $6 \leqslant n \leqslant 7$ and let $M \subset X$ be a twosided compact connected **incompressible hypersurface** which does not admit a metric of $\mathrm{scal} > 0$. Suppose*

- *either X and M are both almost spin,*
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Then X does not admit a complete metric of $\mathrm{scal} > 0$.

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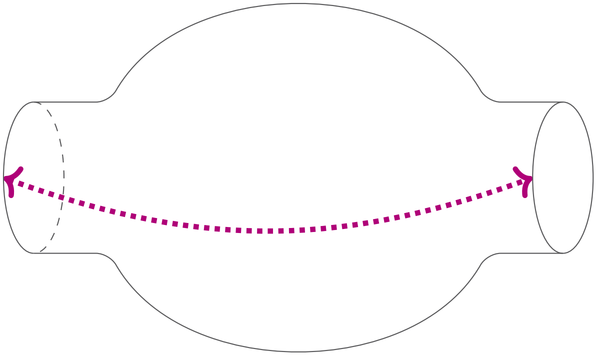
Example

$$M \coloneqq (K3 \times T^2)\#(\mathbb{C}P^2 \times S^2) \text{ is totally non-spin and admits } \mathrm{scal} > 0,$$

but it contains $K3 \times S^1$ as an incompressible hypersurface which is spin and does not admit $\mathrm{scal} > 0$.



Thank you for your attention!