Non-perturbative results of a just-renormalisable model

based on works with Harald Grosse & Raimar Wulkenhaar,
and work in progress with J. Thürigen

Alexander Hock

University of Oxford

alexander.hock@maths.ox.ac.uk

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Overview

- Motivating noncommutative geometry
- ϕ^4 matrix model
- **Double expansion** (in $\frac{1}{N}$ and λ)
- **Exact** solution at genus 0
- Renormalisation Hopf algebra

Why noncommutative space?

- QFT uses operator-valued distributions **smeared** over the support of a test function
- Taking Einstein gravity into account we find a **minimal length**, Planck length
  $\rightarrow$ measure uncertainties at Planck scale
- Uncertainties give rise to noncommutativity
  (e.g. $\Delta x_1 \Delta p_{x_1} \geq \hbar/2 \quad \rightarrow \quad [Q_{x_1}, P_{x_1}] = i\hbar$)
- $[x_1, x_2] = iV \in i\mathbb{R}$
  $\rightarrow$ **Moyal space**

Scalar QFT on the Moyal Space

The action of the **noncommutative real scalar ϕ_D^4 QFT** on the Moyal space is defined by

$$S[\phi] := \frac{1}{8\pi}\int_{\mathbb{R}^D} dx \left(\frac{1}{2}\phi\left(-\Delta + \Omega^2\|2\Theta^{-1}x\|^2 + \mu^2\right)\star\phi + \frac{\lambda}{4}\phi^{\star,4}\right)(x),$$

where Δ is the **Laplacian**, μ the **mass**, λ the **coupling constant** and $\Omega \in \mathbb{R}$. The **Moyal $\star$-product** is defined by

$$(g\star h)(x) = \int_{\mathbb{R}^D}\frac{dk}{(2\pi)^D}\int_{\mathbb{R}^D} dy\, g(x+\tfrac{1}{2}\Theta k)\, h(x+y)e^{\mathrm{i}k\cdot y},$$

$$\Theta = \mathrm{id}_{D/2}\otimes\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\cdot 4V^{2/D}, \quad V\in\mathbb{R}, \quad x\in\mathbb{R}^D, \quad g,h\in\mathcal{S}(\mathbb{R}^D).$$

Notice, $(g\star h)(x) = g(x)h(x)$ for $V=0$

Matrix Base

The **Moyal algebra** $\mathcal{A}_\star = (\mathcal{S}(\mathbb{R}^D), \star)$ is a vector space equipped with the $\star$-product. For this vector space, a **matrix basis** $f_{nm}(x)$ exists with:

$$(f_{nm} \star f_{kl})(x) = \delta_{m,k} f_{nl}(x), \qquad \int_{\mathbb{R}^D} dx\, f_{nm}(x) = 8\pi V \delta_{n,m}.$$

A function $\phi \in \mathcal{C}_0(\mathbb{R}^D)$ that vanishes at infinity can be expanded in this basis

$$\phi(x) = \sum_{n,m} \phi_{nm} f_{nm}(x),$$

where (ϕ_{nm}) is **Hermitian**

Action in the Matrix Base at $\Omega = 1$

Taking renormalization into account, the **renormalized action** is then

$$S[\phi] = V\left(\sum_{n,m} E_n Z \phi_{nm}\phi_{mn} + \frac{Z^2\lambda_{bare}}{4}\sum_{n_i}\phi_{n_1 n_2}\phi_{n_2 n_3}\dots\phi_{n_4 n_1}\right)$$

$$E_n := \frac{\mu_{bare}^2}{2} + \frac{n}{V^{2/D}} \quad \textbf{eigenvalues of the Laplacian}$$

with renormalizations for mass μ_{bare}^2, field Z and coupling constant λ_{bare}.
Important: The eigenvalues E_n have **multiplicities** r_n depending on the **dimension** D

$D = 2 \to r_n = 1, \quad D = 4 \to r_n = n, \quad D = 6 \to r_n = n(n+1)/2$

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Fact

The interaction is **only cyclic symmetric**
$\to$ **oriented Feynman graphs (ribbon graphs)**
$\to$ embedded into **Riemann surfaces** with a genus and boundaries

ϕ^4 Matrix Model and Correlation Functions

Let H_N be the space of Hermitian $(N \times N)$-matrices, $E \in H_N$ positive with eigenvalues (E_n) (from the Laplacian).

The size N of the matrix is **related** to the noncommutativity V.

Define the **partition function**

$$\mathcal{Z} = \int_{H_N} d\phi \exp\left[-S[\phi] \right].$$

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The **2-point correlation function** is by definition

$$G_{pq} := V\langle\phi_{pq}\phi_{qp}\rangle = \frac{V\int_{H_N} d\phi\, \phi_{pq}\phi_{qp} \exp\big[-S[\phi]\big]}{\int_{H_N} d\phi \exp\big[-S[\phi]\big]}$$

for $E_p \neq E_q$.

Computational steps

- Calculating **Dyson-Schwinger equations** (DSE)
- Calculating **Ward-Takahashi identities**
  $\rightarrow$ in the **formal $\frac{1}{N}$-expansion** the DSE **decouple**

$$G_{pq} = \sum_{g=0}^{\infty} N^{-2g} G_{pq}^{(g)}$$

- To recover correlation functions on the Moyal space, the **continuum limit** is performed
- The size of the matrices N and the deformation V tends to ∞ with constant ratio $\frac{N}{V^{2/D}}$ defining the **UV cut-off** Λ^2
- The correlation functions become **continuous functions** on $[0, \Lambda^2]$. For instance, the 2-point function gets

$$\lim_{\substack{V,N\to\infty \\ \frac{N}{V^{2/D}}=\Lambda^2}} G_{pq}^{(g)} =: G^{(g)}(x, y),$$

where $x = \lim \frac{p}{V^{2/D}}$ and $y = \lim \frac{q}{V^{2/D}}$.

Renormalized 2-Point Dyson-Schwinger Equation

The planar 2-point function obeys in a formal N expansion the **nonlinear equation**

$$\left(\mu_{bare}^2 + \frac{p}{V^{2/D}} + \frac{q}{V^{2/D}} + \frac{\lambda_{bare}}{V}\sum_m r_m Z G_{pm}^{(0)}\right) Z G_{pq}^{(0)} = 1 + \frac{\lambda_{bare}}{V}\sum_m r_m Z \frac{G_{mq}^{(0)} - G_{pq}^{(0)}}{\frac{m}{V^{2/D}} - \frac{p}{V^{2/D}}}.$$

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Performing the **continuum limit** (by the scaling limit with constant ratio $\lim \frac{N}{V^{2/D}} \mapsto \Lambda^2$) with $\frac{p}{V^{2/D}} \mapsto x \in [0, \Lambda^2]$ and $G_{pq} \mapsto G(x, y)$ yields

$$\Big(y + \mu_{bare}^2 + x + \lambda_{bare}\int_0^{\Lambda^2} dt\, t^{D/2-1}\, \big(ZG^{(0)}(x,t) + \frac{1}{t-x}\big)\Big) ZG^{(0)}(x,y) = 1 + \lambda_{bare} Z \int_0^{\Lambda^2} dt\, t^{D/2-1} \frac{G^{(0)}(t,y)}{t-x},$$

where $\mu_{bare}, \lambda_{bare}, Z$ depend on Λ^2, the **cut-off**.

The Starting Point

Thm (Grosse, AH, Wulkenhaar '19)

The red part is UV finite and given by

$$\mu_{bare}^2 + x + \lambda_{bare} Z \int_0^{\Lambda^2} dt\, t^{D/2-1} \left(G(x,t) + \frac{1}{t-x} \right) = -R(-R^{-1}(x))$$

where $R(z)$ satisfies

$$R(z) = z - (-z)^{\frac{D}{2}} \lambda \int_0^\infty \frac{dt\, \varrho_\lambda(t)}{(t+\mu^2)^{\frac{D}{2}}(t+\mu^2+z)},$$
$$\varrho_\lambda(t) = R(t)^{D/2-1},$$

Examples

$D=2:$ $$R(z) = -\frac{1}{2} + z + \lambda \log\left(\frac{1}{2}+z\right)$$ (Panzer,Wulkenhaar '18)

$D=4:$ $$R(z) = \Big(-\frac{\mu^2}{2}+z\Big)\, {}_2F_1\Big(\begin{matrix}\alpha_\lambda, & 1-\alpha_\lambda \\ & 2\end{matrix}\Big|\frac{1}{2}-\frac{z}{\mu^2}\Big), \quad \alpha_\lambda = \frac{\arcsin(\lambda\pi)}{\pi}$$

finite N : $$R(z) = z - \frac{\lambda}{V}\sum_{k=1}^{N} \frac{r_k}{R'(\varepsilon_k)(z+\varepsilon_k)}, \quad E_n = R(\varepsilon_n), \quad \lim_{\lambda\to 0} \varepsilon_n = E_n$$

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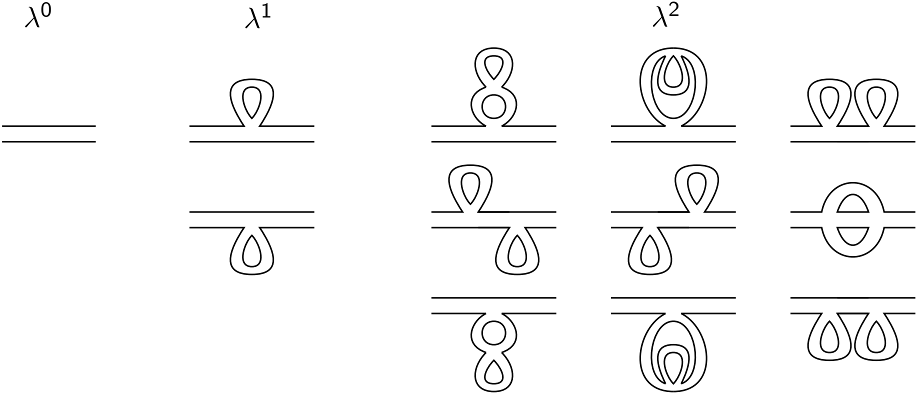
Finite radius of convergence in λ!!

In 4D: Exact Solution of the 2-Point Function

Solving the **singular integral equation** (of Carleman type) yields
$G^{(0)}(x,y) = \frac{\mu^2 \exp(N(x,y))}{\mu^2+x+y}$

$$
\begin{aligned}
N(x,y) := \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} dt \Big\{ & \log\big(x - R(-\tfrac{\mu^2}{2} - \mathrm{i}t)\big) \frac{d}{dt} \log\big(y - R(-\tfrac{\mu^2}{2} + \mathrm{i}t)\big) \\
& - \log\big(- R(-\tfrac{\mu^2}{2} - \mathrm{i}t)\big) \frac{d}{dt} \log\big(- R(-\tfrac{\mu^2}{2} + \mathrm{i}t)\big) \\
& - \log\big(x - (-\tfrac{\mu^2}{2} - \mathrm{i}t)\big) \frac{d}{dt} \log\big(y - (-\tfrac{\mu^2}{2} + \mathrm{i}t)\big) \\
& + \log\big(- (-\tfrac{\mu^2}{2} - \mathrm{i}t)\big) \frac{d}{dt} \log\big(- (-\tfrac{\mu^2}{2} + \mathrm{i}t)\big) \Big\},
\end{aligned}
$$

Graph Expansion of the 2-Point Function $G^{(0)}(x, y)$



Exact result coincides with the perturbative expansion!

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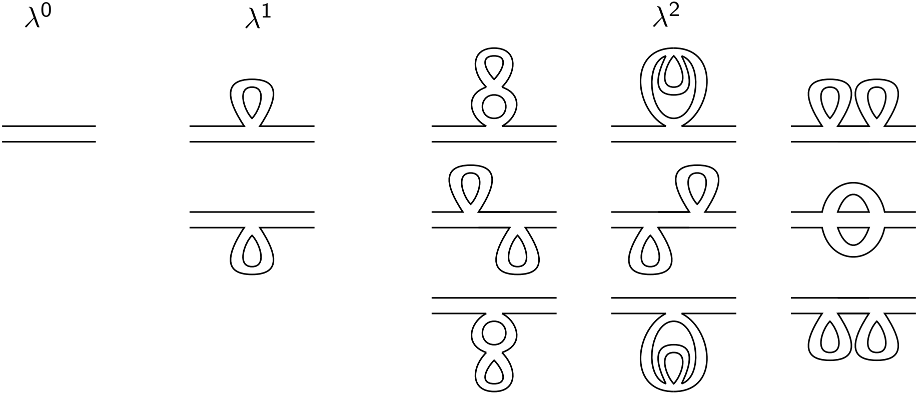

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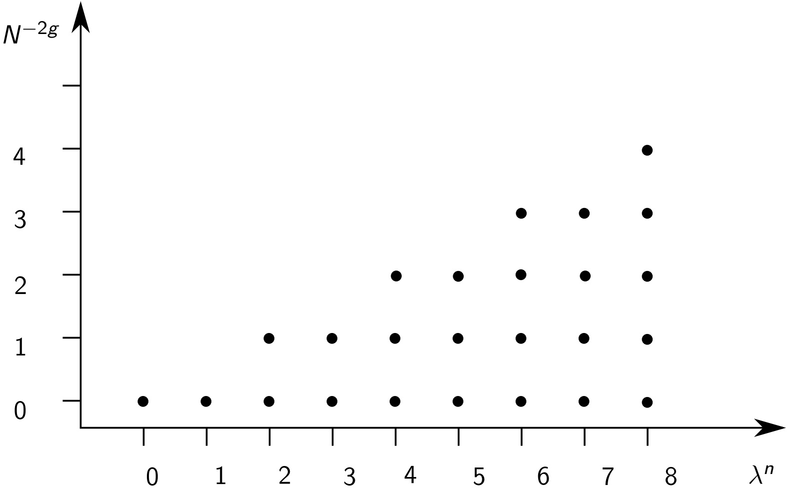
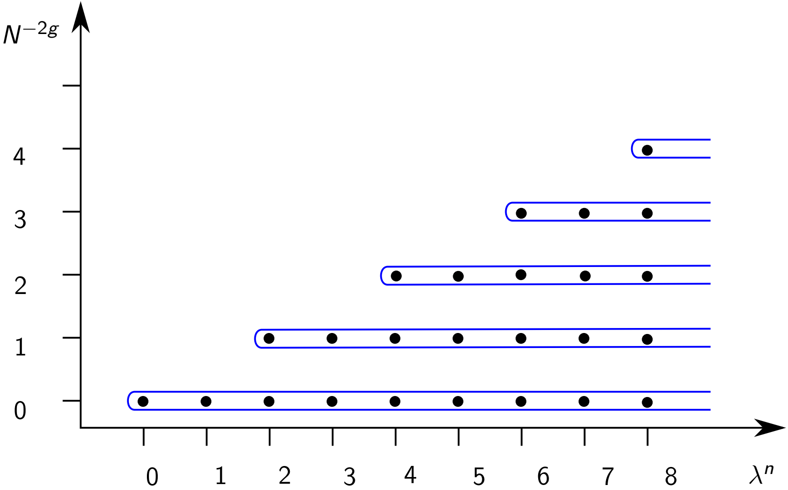
Renormalon problem from perturbation theory


N^{-2g}
4
3
2
1
0
0
1
2
3
4
5
6
7
8
λ^n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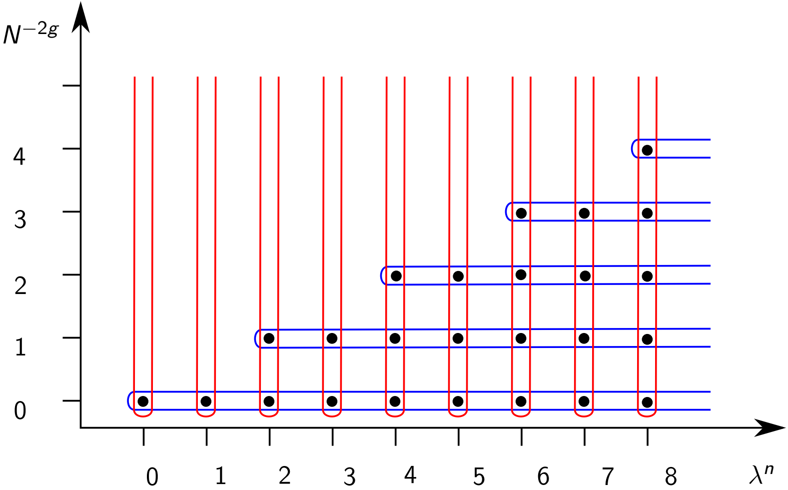

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Spectral Dimension of ϕ_4^4

The asymptotic of the hypergeometric functions

$$ {}_2F_1\left(\begin{matrix} a, & 1-a \\ & 2 \end{matrix}\,\middle|\, -x\right) \overset{x\to\infty}{\sim} \frac{1}{x^a}. $$

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The R-function defines an **effective measure**, which behaves asymptotically

$$ R(x) = x\,{}_2F_1\left(\begin{matrix} \alpha_\lambda,\ 1-\alpha_\lambda \\ 2 \end{matrix}\,\middle|\, -\frac{x}{\mu^2}\right) \overset{x\to\infty}{\sim} x^{1-\alpha_\lambda}, $$

where $\alpha_\lambda = \frac{\arcsin(\lambda\pi)}{\pi}$.
Finally, the **spectral dimension** D has the asymptotics
$x^{\frac{D}{2}-1} \quad \rightarrow \quad D = 4 - 2\frac{\arcsin(\lambda\pi)}{\pi}.$

Why does it avoid the Triviality Problem?

The inverse R^{-1} is an essential ingredient for the exact solution!
Would instead the solution be constructed by

$$\tilde{R}(x) = x - \lambda x^2 \int_0^\infty \frac{d\varrho_0(t)}{(\mu^2 + t)^2(\mu^2 + t + x)}, \qquad d\varrho_0(t) = dt\, t$$

$\Rightarrow$ no inverse exists globally on $\mathbb{R}_+$
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The function $R(x)$ has a global inverse on $\mathbb{R}_+$!
The effective dimension drop is only **visible** on the level of exact solutions
Not accessible with perturbation theory!

Perturbative Renormalisation

Question

Since we can **resum each genus sector** in λ, does that mean that **perturbative renormalisation is simpler**?

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Looking for example at the sunrise graph, we have **perturbatively "overlapping" divergencies**.

Subtracting subdivergences in QFT correctly is described via **BPHZ forest formula**.

The BPHZ forest formula has (secretely) a **Hopf algebra structure**.

This was revealed by **Allain Connes** and **Dirk Kreimer**.

Does the Hopf algebraic structure of the model under consideration differ from the other 4D QFTs?

Hopf algebra of ribbon graphs [J. Thürigen]

For a fixed external structure (e.g. 2-point and 4-point graphs of genus $g = 0$), let G be the set of all connected 1PI ribbon graphs. Then, the $\mathbb{Q}$-algebra generated by G

$$\mathcal{G} = \langle \mathsf{G} \rangle$$

is a Hopf algebra.

Product is the disjoint union
Co-product $\Delta : \mathsf{G} \to \mathsf{G} \otimes \mathsf{G}, \Gamma \mapsto \sum_{\Theta} \Theta \otimes \Gamma/\Theta$, where Γ/Θ contracts Θ in Γ, e.g.

Series of 1PI graphs

Consider the **series**

$$X^{\gamma} = 1 \pm \sum_{\substack{\Gamma \in \mathrm{G} \\ res(\Gamma) = \gamma}} \alpha^{F_\Gamma} \frac{\Gamma}{|Aut\Gamma|} = 1 \pm \sum_{j=1}^{\infty} \alpha^j c_j^{\gamma},$$

where γ is a fixed external structure, i.e. $\Gamma/\Gamma = \gamma$.

For the 1PI 2-point function and 4-point we abbriviate

$$X_2 \equiv X^{-}, \qquad X_4 \equiv X^{\times},$$

where $\gamma = -$ is the external structure of the 2-point function and $\gamma = \times$ of the 4-point.

Combinatorial Dyson-Schwinger equations

$$X_2 = \mathbb{1} - \alpha B^2\left(\frac{X_4}{X_2}\right) = \mathbb{1} - \alpha(B$$  $$+ B$$  $$)\left(\frac{X_4}{X_2}\right),$$

$$X_4 = 1 + \sum_{\substack{\text{over all primitive}\\ \text{4-point graphs } \Gamma}} \alpha^{F_\Gamma} B^\Gamma\left(\left(\frac{X_4}{X_2^2}\right)^{F_\Gamma} X_4\right)$$

$$= 1 + \alpha(B$$  $$+ B$$  $$)\left(\frac{X_4^2}{X_2^2}\right) + \ldots,$$

Primitive graphs have no subdivergencies. **Grafting operator** B^γ

$$B^\gamma(X) = \frac{1}{(\gamma|X)|X|_\vee} \sum_{\substack{\Gamma \in G\\ res(\Gamma)=res(\gamma)}} \frac{\text{bij}(\gamma, X, \Gamma)}{\text{maxf}(\Gamma)}\Gamma,$$

symmetry factors $(\gamma|X), |X|_\vee, \text{maxf}(\Gamma), \text{bij}(\gamma, X, \Gamma)$ such that the combinatorial DSEs hold **by definition**.

Observations/Properties of Hopf algebra structure

- **coupled system** of combinatorial DSE
- **infinitely many** primitive 4-point graphs
- c_n^γ generate a **Hopf subalgebra**, i.e. $\Delta(c_n^\gamma) = \sum_{k=0}^n P_{n,k}^\gamma(c) \otimes c_{n-k}^\gamma$
- the grafting operator is Hochschild 1-cocycle, i.e.

$$\Delta B = B \otimes 1 + (1 \otimes B)\Delta.$$

- an algebra homomorphism to the space a Feynman amplitudes is very **complicated**, we have to include all the **symmetry factors** of the grafting operator B
  $\rightarrow$ exact results are **not accessible** from the Hopf algebraic perspective

Summary

- Use **noncommutative geometry** to combine **gravity and QFT**
- **matrix model** with nontrivial covariance
- **Closed Dyson-Schwinger equations** after using Ward identites in the **formal $\frac{1}{N}$ expansion**
- **exact solution in λ at each order in $\frac{1}{N}$**
- **effective drop** of the spectral dimension to $D = 4 - 2\frac{\arcsin(\lambda\pi)}{\pi}$
- **resummation** in N is not absolutely convergent (**resurgence**)
- at a fixed order in λ **finitely many terms contribute**, resummable in N at a finite order in λ
- just the **genus $g = 0$ sector** has to be **renormalised**
- $g = 0$ has the **renormalisation complexity of an ordinary QFT** $\rightarrow$ both in the sense of **field and mass renormalisation** Z, μ_{bare} and from **Hopf algebraic** perspective (except for running coupling in λ)