Phase Transitions in Quantum Many-body Theory

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Harbin Institute of Technology

From Perturbative to non-Perturbative QFT
Münster University

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The Content

- The 2-D Honeycomb-Hubbard model and main results.

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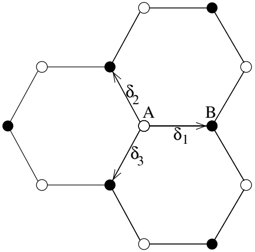
- Sketch of the proof: fermionic cluster expansions and renormalization group analysis.
- Hubbard model on the 2-D square lattice and universality.
- Conclusions and perspectives.

The 2d Honeycomb Hubbard model



- The honeycomb lattice $\Lambda = \Lambda^A \cup \Lambda^B$ is the superposition of the triangular lattice Λ^A (White dots) with $\Lambda^B = \Lambda^A + \vec{\delta}_i$ (Black dots): $\vec{\delta}_1 = (1, 0)$, $\vec{\delta}_2 = \frac{1}{2}(-1, \sqrt{3})$, $\vec{\delta}_3 = \frac{1}{2}(-1, -\sqrt{3})$.

The states of the system.

- Let $\Lambda_L = \Lambda / L\Lambda$, $L \in \mathbb{N}$. The one-particle Hilbert space $\mathcal{H}_L = \{\psi_{\mathbf{x},\alpha,\tau} : \Lambda_L \times \{A, B\} \times \{\uparrow, \downarrow\} \to \mathbb{C}\ \}$ such that $\|\psi\|_2^2 = \sum_{\mathbf{x},\tau,\alpha} |\psi_{\mathbf{x},\alpha,\tau}|^2 = 1$.

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- The Fermionic Fock space $\mathcal{F}_L$ over $\mathcal{H}_L$:

$$\mathcal{F}_L = \mathbb{C} \oplus \bigoplus_{N=1}^{4L^2} \mathcal{F}_\Lambda^{(N)}, \quad \mathcal{F}_L^{(N)} = \bigwedge^N \mathcal{H}_L.$$

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- For any $\psi \in \mathcal{H}_L$, we can define the Fermionic operators $a^\pm(\psi)$ (while dots) and $b^\pm(\psi)$ (black dots) satisfying the CAR:

$$\begin{aligned}
\{a^+(\psi), a^-(\phi)\} &:= a^+(\psi)a^-(\phi) + a^-(\phi)a^+(\psi) \\
&= \langle \psi, \phi \rangle_{\mathcal{H}_L} \\
\{a^+(\psi), a^+(\phi)\} &= 0 = \{a^-(\psi), a^-(\phi)\}
\end{aligned}$$

The Fermionic operators

- The operators $a^{\pm}(\psi)$ (while dots) and $b^{\pm}(\psi)$ (black dots) acting on $\mathcal{F}_L$, $(\xi = (\mathbf{x}, \tau))$ by:

$$
\begin{aligned}
&(a^{+}(\psi)\Psi)^{(N)}(\xi_1, \cdots, \xi_N) \\
&= \sum_{j=1}^{N} \frac{(-1)^j}{\sqrt{N}} \psi(\xi_j)\psi^{(N-1)}(\xi_1, \cdots, \xi_{j-1}, \xi_{j+1}, \cdots \xi_N), \\
&(a^{-}(\psi)\Psi)^{(N)}(\xi_1, \cdots, \xi_N) \\
&= \sqrt{N+1} \int d\xi \bar{\psi}(\xi)\psi^{(N+1)}(\xi, \xi_1, \cdots, \xi_N)
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- The CAR for $\{a^{\pm}_{\mathbf{x},\tau}\}$: $\{a^{+}_{\mathbf{x},\tau}, a^{-}_{\mathbf{x}',\tau'}\} = \delta_{\mathbf{x},\mathbf{x}'}\delta_{\tau,\tau'}$, $\{a^{+}_{\mathbf{x},\tau}, a^{+}_{\mathbf{x}',\tau'}\} = 0$, $\{a^{-}_{\mathbf{x},\tau}, a^{-}_{\mathbf{x}',\tau'}\} = 0$. The same for $b^{\pm}_{\mathbf{z},\tau}$.

The Hubbard model on the honeycomb lattice

The grand-canonical Hamiltonian is:

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\begin{aligned}
H_{\Lambda_L} &= -t \sum_{\substack{\mathbf{x}\in\Lambda_A \\ i=1,2,3}} \sum_{\tau=\uparrow\downarrow} \left(a^+_{\mathbf{x},\tau} b^-_{\mathbf{x}+\vec{\delta}_i,\tau} + b^+_{\mathbf{x}+\vec{\delta}_i,\tau} a^-_{\mathbf{x},\tau} \right) \\
&- \mu \sum_{\mathbf{x}\in\Lambda_A} \sum_{\tau=\uparrow\downarrow} \left(a^+_{\mathbf{x},\tau} a^-_{\mathbf{x},\tau} + b^+_{\mathbf{x}+\vec{\delta}_i,\tau} b^-_{\mathbf{x}+\vec{\delta}_i,\tau} \right) \\
&+ \lambda \sum_{\mathbf{x}\in\Lambda_A} \left(a^+_{\mathbf{x},\uparrow} a^-_{\mathbf{x},\uparrow} a^+_{\mathbf{x},\downarrow} a^-_{\mathbf{x},\downarrow} + b^+_{\mathbf{x},\uparrow} b^-_{\mathbf{x},\uparrow} b^+_{\mathbf{x},\downarrow} b^-_{\mathbf{x},\downarrow} \right)
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- $\mathbf{x}$, coordinates of the sites, $\tau =\uparrow\downarrow$ are the spins.
- When $\lambda = 0$, any fermion is only hopping to its nearest neighbor. When $\lambda > 0$, all fermions are correlated through the interaction term.

(Imaginary)-Time evolution and the correlation functions

- Let $\mathbf{a}^{\pm}_{\mathbf{x},1} = \mathbf{a}^{\pm}_{\mathbf{x}}$, $\mathbf{a}^{\pm}_{\mathbf{x},2} = \mathbf{b}^{\pm}_{\mathbf{x}}$. Define the imaginary-time evolution: $\mathbf{a}^{\pm}_{x,\alpha} = e^{H_{\Lambda_L} x^0} \mathbf{a}^{\pm}_{\mathbf{x},\alpha} e^{-H_{\Lambda_L} x^0}$, $x = (x^0, \mathbf{x}) \in \Lambda_{\beta,L} := [-\beta, \beta) \times \Lambda_L$, $\beta = 1/T$.

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- The Gibbs state associated with the Hamiltonian H_{Λ_L} is: $\langle \cdot \rangle = \mathrm{Tr}_{\mathcal{F}_L} [\cdot \, e^{-\beta H_{\Lambda_L}}]/Z_{\beta,\Lambda_L}$, $Z_{\beta,\Lambda_L} = \mathrm{Tr}_{\mathcal{F}_L} e^{-\beta H_{\Lambda_L}}$.

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- Interesting quantities are:
  - The $2p$-point Schwinger's function $p \geq 0$ ($2p$-th moments of the Gibbs states) for $L \to \infty$: $[S_{2,\beta}(x_1, x_2, \lambda)]_{\alpha_1,\alpha_2} = \lim_{L\to\infty} \langle \mathbf{T} \mathbf{a}^{\varepsilon_1}_{x_1,\alpha_1,\tau_1} \mathbf{a}^{\varepsilon_2}_{x_2,\alpha_2,\tau_2} \rangle_{\beta,L}$ $\langle \cdot \rangle = \mathrm{Tr}_{\mathcal{F}_L} [\, \cdot \, e^{-\beta H_{\Lambda_L}}]/Z_{\beta,\Lambda_L}$, $\mathbf{T}$ is the time-ordering operator.

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  - The connected Schwinger's function $S^c_{2,\beta}(x_1, x_2, \lambda)$ "cummulants of the Gibbs state" and the self-energy $\Sigma_{2,\beta}(x_1, x_2; \lambda)$.

The noninteracting two-point Schwinger function ($\lambda = 0$)

- $C(x_1, x_2, 0, \mu) = \int dk_0 d\mathbf{k} \hat{C}(k_0, \mathbf{k}, 0) e^{ik(x_1 - x_2)}$,
  $k_0 = (2n+1)\pi T$, $n \in \mathbb{Z}_{\geq 0}$, $\mathbf{k} = (k_1, k_2) \in \mathcal{B} = \mathbb{R}^2 / \Lambda^*$,
  $\Lambda^* = \{\mathbf{k} \in \mathbb{R}^2, \langle \mathbf{x}, \mathbf{k} \rangle \in 2\pi\mathbb{Z},\ \forall\ \mathbf{x} \in \Lambda\}$.

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$$\hat{C}(k_0, \mathbf{k}, 0) = \frac{1}{k_0^2 + |\Omega(\mathbf{k})|^2 - \mu^2 - 2i\mu k_0} \begin{pmatrix} ik_0 + \mu & -\Omega^*(\mathbf{k}) \\ \Omega(\mathbf{k}) & ik_0 + \mu \end{pmatrix}$$

$$\Omega(\mathbf{k}) = 1 + 2e^{-i\frac{3}{2}k_1} \cos(\tfrac{\sqrt{3}}{2} k_2)$$

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- It is well defined for $T > 0$ ($k_0 > 0$).
- For $T \to 0$, $\hat{S}_{2,\beta}(k_0, \mathbf{k}, 0)$ is singular on the set:
  $\mathcal{F}_0 = \{\mathbf{k} \in \mathcal{B}, |\Omega(\mathbf{k})| - \mu = 0\}$, called the Fermi surface.

The Fermi surfaces

- When $\mu = 0$, $\mathcal{F}_0 = \{\mathbf{k}_F^{\pm} = (\frac{2\pi}{3}, \pm\frac{2\pi}{3\sqrt{3}})\}$ is a pair of points.

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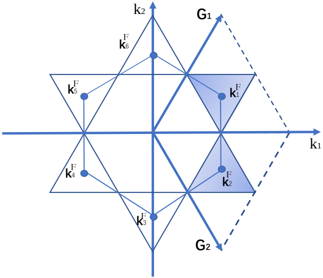

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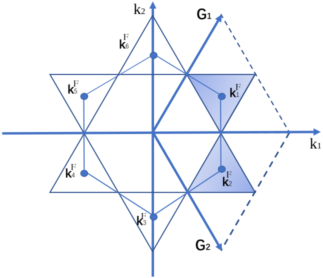

- For $0 < \mu < 1$, $\mathcal{F}_0$ is a set of convex curves surrounding $\mathbf{k}_F^{\pm}$.

The interacting theory $\lambda \neq 0$

The fundamental questions are:

- Is $\lim_{L\to\infty} \frac{Z_{\beta,\Lambda_L}(\lambda)}{Z_{\beta,\Lambda_L}(0)}$ or $\lim_{L\to\infty} \log Z_{\beta,\Lambda_L}(\lambda)$ a well-defined quantity? Or can we rigorously define this model?

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- The analytic properties of the (connected)-Schwinger functions?
- Is the ground state of this model a Fermi liquid?

The Fermi liquid

Definition (Fermi liquid, Salmhofer, 1998)

Let $\hat{S}^c_{2,\beta}(k,\lambda)$ be the Fourier transform of $S^c_{2,\beta}(x_1,x_2,\lambda)$. The ground state of an interacting many-fermion system is said to be a Fermi liquid in the equilibrium (at $\beta = 1/T$) if

- $\hat{S}_{2,\beta}(k,\lambda)$ is an analytic function of the coupling constant λ for $\beta < \infty$.

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- $\hat{S}_{2,\beta}(k,\lambda)$ is an analytic function of the coupling constant λ for $\beta < \infty$.
- The Fourier transform of the self-energy function, $\hat{\Sigma}(k,\lambda,\beta)$, is C^2 in k for $\beta \to \infty$.

Examples of Fermi liquids and non-Fermi liquids

- Jellium model: FL for $T \geq T_c$ (Disertori, Rivasseau 2000)
- Fermic model in the continuum with central symmetric Fermi surfaces: FL for $T \geq T_c$. (Benfatto, Giuliani, Mastropietro 2003)
- Many-fermion model with asymmetric Fermi surfaces: FL for $T \to 0$: (Feldman, Knörrer, Trubowitz 2004)
- Hubbard model on the square lattice at half-filling: **Non-fermi liquid** for $T \geq T_c$ (Afchain, Magnen, Rivasseau 2005)
- Hubbard model on the square lattice far from half-filling: FL for $T \geq T_c$ (Benfatto, Giuliani, Mastropietro 2006)

The Honeycomb Hubbard model with $\mu = 0$, $\lambda \neq 0$. (Graphene)

The Honeycomb Hubbard model with $\mu = 0$, $\lambda \neq 0$. (Graphene)

- Theorem (Giuliani, Mastropietro, 2010)
  There exists a positive constant U such that the "pressure function" $\log \frac{Z_{\beta,\Lambda}(\lambda)}{Z_{\beta,\Lambda}(0)}$ *and the connected Schwinger function* $S^c_{2,\beta}(x_1, x_2, \lambda)$ *are both analytic functions of* λ *when* $\beta \to \infty$*, for* $|\lambda| \leq U$.

The Honeycomb Hubbard model with $\mu = 1$, $\lambda \neq 0$

Theorem (Rivasseau, ZW 2021)

- *There exists a positive constants* $\beta_c = 1/T_c$ *such that for any* $\beta \leq \beta_c$, *the "pressure function"* $\log \frac{Z_{\beta,\Lambda}(\lambda)}{Z_{\beta,\Lambda}(0)}$ *and the connected two-point function* $S^c_{2,\beta}(\lambda)$ *are analytic functions of the coupling constant* λ, *in the region*

$$|\lambda \log^2 \beta| < 1. \tag{1}$$

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- *Fix λ, with $|\lambda| < 1$, the transition temperature is $T_c = C_1 e^{-\frac{C_2}{|\lambda|^{1/2}}}$, $C_1, C_2 > 0$ are two strictly positive constants.*

The Honeycomb Hubbard model with $\mu = 1$, $\lambda \neq 0$

Theorem (Rivasseau, ZW 2021)

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- *The self-energy function $\hat{\Sigma}(k,\lambda)$ is $C^{1+\epsilon}$ differentiable w.r.t. the momentum for $T \to 0$. The ground state is not a Fermi liquid.*

Proof-The Grassmann algebra and Berezin Integrals

- The Grassmann algebra **Gra** is an associative, non-commutative, nilpotent algebra generated by the Grassmann variables $\{\hat{\psi}^{\varepsilon}_{k,\alpha}\}$, $\varepsilon = \pm$, $\alpha = 1, 2$, $k = (k_0, \mathbf{k}) \in \mathcal{D}_{\beta,L} = \{\frac{2\pi}{\beta}(n + \frac{1}{2}), n \in \mathbb{N}\} \times \mathcal{D}_L$, $\mathcal{D}_L = \mathbb{R}^2/\Lambda^*_L$ such that $\hat{\psi}^{\varepsilon}_{k,\alpha}\hat{\psi}^{\varepsilon'}_{k',\alpha'} = -\hat{\psi}^{\varepsilon'}_{k',\alpha'}\hat{\psi}^{\varepsilon}_{k,\alpha}$ and $(\hat{\psi}^{\varepsilon}_{k,\alpha})^2 = 0$.

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- A model of **Gra** is the exterior algebra $(dx, \wedge)$

The Berezin Integrals

The Grassmann Gaussian measure $P(d\psi)$ with covariance $\hat{C}(k)$:

$$P(d\psi) = N^{-1} D\psi \cdot \exp\left\{ -\frac{1}{\beta|\Lambda_L|} \sum_{k=(k_0,\mathbf{k})\in\mathcal{D}_{\beta,L},\tau,\alpha} \hat{\psi}^+_{k,\tau,\alpha} \hat{C}(k)^{-1} \hat{\psi}^-_{k,\tau,\alpha} \right\}$$

where $N = \prod_{\mathbf{k}\in\mathcal{D}_L,\tau=\uparrow\downarrow} \frac{1}{\beta|\Lambda_L|} \begin{pmatrix} -ik_0 - 1 & -\Omega^*(\mathbf{k}) \\ -\Omega(\mathbf{k}) & -ik_0 - 1 \end{pmatrix}$,

$$\lim_{L\to\infty} \int P(d\psi) \hat{\psi}^-_{k_1,\tau_1,\alpha_1} \hat{\psi}^+_{k_2,\tau_2,\alpha_2} = \delta_{k_1,k_2} \delta_{\tau_1,\tau_2} [\hat{C}(k_1)]_{\alpha_1,\alpha_2}.$$

The Berezin integrals

- Define $\psi^{\pm}_{x,\tau,\alpha} = \frac{1}{\beta|\Lambda_L|}\sum_{k\in\mathcal{D}_{\beta,L}} e^{\pm ikx}\hat{\psi}^{\pm}_{k,\tau,\alpha}$, $x \in \Lambda_{\beta,L} := [-\beta,\beta) \times \Lambda_L$,

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- the interacting potential becomes:

$$\mathcal{V}(\psi) = \lambda \sum_{\alpha,\alpha'=1,2} \int_{\Lambda_{\beta,L}} d^3x\, \psi^{+}_{x,\uparrow,\alpha}\psi^{-}_{x,\uparrow,\alpha'}\psi^{+}_{x,\downarrow,\alpha}\psi^{-}_{x,\downarrow,\alpha'}\ ,$$

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- The Schwinger functions:

$$S_{n,\beta}(x_1,\cdots,x_n) = \lim_{L\to\infty}\frac{1}{Z_L}\int \psi^{\epsilon_1}_{x_1,\tau_1,\alpha_1}\cdots\psi^{\epsilon_n}_{x_n,\tau_n,\alpha_n} e^{-\lambda\mathcal{V}(\psi)}P(d\psi).$$

Generating functionals

- Let j^+, j^- be two Grassmann variables. Define: $Z(j^+, j^-, \lambda) = \int e^{-\lambda \mathcal{V}(\psi) + \int dx \psi^+(x) j^-(x) + + \int dx j^+(x) \psi^-(x)} P(d\psi)$. and $W(j^\pm, \lambda) = \log Z(j^\pm, \lambda)$, the cumulant generating functional.

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- The connected $2p$-point Schwinger's functions:

$$S^c_{2p}(x_1, \cdots, x_p, y_1 \cdots, y_p) = \prod_{i=1}^{p} \frac{\delta^2}{\delta j^+(x_i) \delta j^-(y_i)} W(j^+, j^-)|_{j^\pm = 0}$$

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- Define $\phi^+(x) = \frac{\delta}{\delta j^-(x)} W(j)$, $\phi^-(x) = \frac{\delta}{\delta j^+(x)} W(j)$, define $\Gamma(\phi^+, \phi^-, \lambda) = W(j^+, j^-, \lambda) - \int d^3x \, [j^+(x) \phi^-(x) + \phi^+(x) j^-(x)]$

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- The self-energy $\Sigma(x, y, \lambda) = \frac{\delta^2}{\delta \phi^+(x) \delta \phi^-(y)} \Gamma(\phi^+, \phi^-, \lambda)|_{\phi^\pm = 0}$

The (naive) perturbation expansion

For $|\lambda| < 1$, perform perturbation expansions:

$$Z(\lambda) = \int P(d\psi) e^{\lambda \int_{\Lambda_{\beta,L}} d^3x \left[\psi^+_{x,\uparrow}\psi^-_{x,\uparrow}\psi^+_{x,\downarrow}\psi^-_{x,\downarrow}\right]}$$

$$" = " \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int P(d\psi) \Big[\int_{\Lambda_\beta} d^3x \, (\psi^+_{x,\uparrow}\psi^-_{x,\uparrow}\psi^+_{x,\downarrow}\psi^-_{x,\downarrow}) \Big]^n .$$

$$= \sum_n \frac{\lambda^n}{n!} \int_{(\Lambda_{\beta,L})^n} d^3x_1 \cdots d^3x_n \begin{Bmatrix} x_{1,\varepsilon_1,\tau_1} & \cdots & x_{n,\varepsilon_n,\tau_n} \\ x_{1,\varepsilon_1,\tau_1} & \cdots & x_{n,\varepsilon_n,\tau_n} \end{Bmatrix},$$

$\{\cdot\}$ is a $2n \times 2n$ determinant, Cayley's notation:

$$\begin{Bmatrix} x_{i,\tau} \\ x_{j,\tau'} \end{Bmatrix} = \det\left[\delta_{\tau\tau'}[\, C(x_i - x_j)]\right], C(x-y) = \int_{\Lambda_{\beta,L}} \hat{C}(k) e^{ik(x-y)} d^3x$$

$$\hat{C}(k) = \frac{1}{k_0^2 + |\Omega(\mathbf{k})|^2 - \mu^2 - 2i\mu k_0} \begin{pmatrix} ik_0 + \mu & -\Omega^*(\mathbf{k}) \\ \Omega(\mathbf{k}) & ik_0 + \mu \end{pmatrix}$$

Difficulties and solutions

- Q1: The perturbation series can be labeled by graphs, called the Feynman graphs. Fully expansion of the determinant generates the combinatorial factor $(2n)!$, which makes the perturbation series divergent.

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- Solution: partially expand the determinant (fermionic cluster expansions) so that only the terms corresponding to spanning forests appear.

The Fermionic cluster expansions for $\log Z$

- Let $\{\mathcal{T}\}$ be the set of spanning trees of G and $w(G, \mathcal{T})$ be a probability measure on $\{\mathcal{T}\}$: $\sum_{\mathcal{T}\subset G} w(G, \mathcal{T}) = 1$.

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- Organizing the perturbation terms as:
  $S = \sum_G A_G = \sum_G \sum_{\mathcal{T}\subset G} w(G,\mathcal{T}) A_G = \sum_{\mathcal{T}} A_{\mathcal{T}}$,
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- The canonical way of defining the weights is the BKAR forest formula (Brydges, Kennedy 87, Abdesselam Rivasseau 95).

Difficulties and solutions

- Q2: $\hat{C}(k)$ is singular for $k_0 \to 0$, $\mathbf{k} \in \mathcal{F}$. Typical term in the perturbation series is $\int dk \cdots [\hat{C}(k)]^p$. But $\hat{C}(k)$ is locally L^1 but not L^p, $\forall p \geq 2$;

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- Solution: The singularities are approached in multi-steps.

The multi-scale analysis

- Let $G_0^h(\mathbb{R})$, $h > 1$, be the Gevrey class of compactly supported functions. Define a cutoff function $\chi \in G_0^h(\mathbb{R})$ as:

$$\chi(t) = \chi(-t) = \begin{cases} = 0 \ , & \text{for} \quad |t| > 2, \\ \in (0,1) \ , & \text{for} \quad 1 < |t| \leq 2, \\ = 1, & \text{for} \quad |t| \leq 1. \end{cases} \tag{2}$$

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- Given fixed constant $\gamma \geq 10$, construct a partition of unity

$$\begin{aligned} 1 &= \sum_{j=0}^{j_{max}} \chi_j(t), \quad j_{max} = E(\log_\gamma \frac{1}{T}); \\ \chi_0(t) &= 1 - \chi(t), \\ \chi_j(t) &= \chi(\gamma^{2j-1} t) - \chi(\gamma^{2j} t) \text{ for } j \geq 1. \end{aligned} \tag{3}$$

The multi-slice expansion

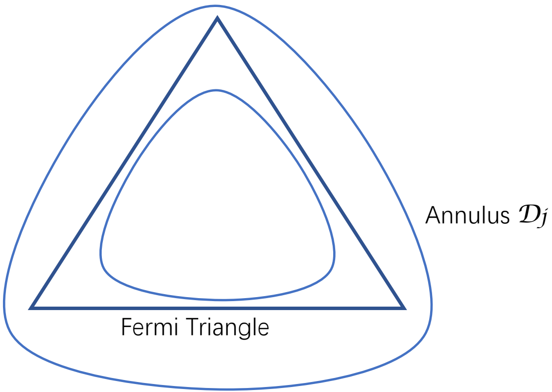
- The free propagator is decomposed as :

$$\hat{C}(k)_{\alpha\alpha'} = \sum_{j=0}^{j_{max}} \hat{C}_j(k)_{\alpha\alpha'}, \ \alpha, \alpha' = 1, 2,$$

$$\hat{C}_j(k)_{\alpha\alpha'} = \hat{C}(k)_{\alpha\alpha'} \cdot \chi_j[4k_0^2 + e^2(\mathbf{k})],$$

$$e(\mathbf{k}) = 8[\cos(\sqrt{3}k_2/2)] \cdot [\cos(\frac{1}{4}(3k_1 + \sqrt{3}k_2))]$$

$$\cdot[\cos(\frac{1}{4}(3k_1 - \sqrt{3}k_2))].$$



Annulus $\mathcal{D}_j$
Fermi Triangle

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- $\hat{C}_j(k) = \sum_{\sigma=(s_a,s_b)} \hat{C}_{j,\sigma}(k)$, $\hat{C}_{j,\sigma}(k) = \hat{C}_j(k) \cdot v_{s_a}[t_a]\ v_{s_b}[t_b]$, $a, b \in \{1,2,3\}$, $t_1 = \cos^2(\sqrt{3}k_2/2)$, $t_2 = \cos^2(\frac{1}{4}(3k_1 + \sqrt{3}k_2))$, $t_3 = \cos^2(\frac{1}{4}(3k_1 - \sqrt{3}k_2))$.

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- $\hat{C}_j(k)=\sum_{\sigma=(s_a,s_b)}\hat{C}_{j,\sigma}(k),\ \hat{C}_{j,\sigma}(k)=\hat{C}_j(k)\cdot v_{s_a}[t_a]\ v_{s_b}[t_b]$, $a,b\in\{1,2,3\},\ t_1=\cos^2(\sqrt{3}k_2/2)$, $t_2=\cos^2(\frac{1}{4}(3k_1+\sqrt{3}k_2)),\ t_3=\cos^2(\frac{1}{4}(3k_1-\sqrt{3}k_2))$.
- Correspondingly, $\hat{\psi}^{\pm,j}_{k,\tau,\alpha}=\sum_{\sigma=(s_a,s_b)}\hat{\psi}^{\pm,j,\sigma}_{k,\tau,\alpha}$, and $\hat{C}_{j,\sigma}(k)$ is the covariance of $\hat{\psi}^{\pm,j,\sigma}_{k,\tau,\alpha}$.

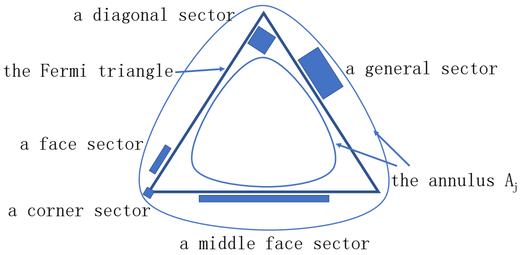

Figure: An illustration of the various sectors.

In each shell of scale $j \in [0, j_{max}]$, a sector is of size $\gamma^{-s_a} \times \gamma^{-s_b}$, in which $0 \leq s_a, s_b \leq j$, $s_a + s_b \geq j - 2$.

The bounds for the propagators

- $$\|C_{j,\sigma}(x-y)]_{\alpha\alpha'}\|_{L^\infty} \le O(1)\gamma^{-s_a-s_b}\ e^{-c[d_{j,\sigma}(x,y)]^{\alpha_0}},$$

  where $0 \le s_a, s_b \le j$, $\alpha_0 = 1/h$, and
  $d_{j,\sigma}(x,y) = \gamma^{-j}|x_0 - y_0| + \gamma^{-s_a}|x_a - y_a| + \gamma^{-s_b}|x_b - y_b|$

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- $$\left\| \, [C_{j,\sigma}(x)]_{\alpha\alpha'} \, \right\|_{L^1} \leq O(1)\gamma^{j}.$$

Theorem (The BKAR jungle Formula. Brydges, Kennedy 87, Abdesselam Rivasseau 95)

Let $I_n = \{1, \cdots, n\}$, $\mathcal{P}_n = \{\ell = (i,j), i,j \in I_n, i \neq j\}$, $\mathcal{S}$ a set of smooth functions from $\mathbb{R}^{\mathcal{P}_n}$ to some Banach space, $\mathbf{1} \in \mathbb{R}^{\mathcal{P}_n}$ be the vector with every entry equals 1. Then for any $\mathbf{x} = (x_\ell)_{\ell \in \mathcal{P}_n} \in \mathbb{R}^{\mathcal{P}_n}$ and $f \in \mathcal{S}$:

$$f(\mathbf{1}) = \sum_{\mathcal{J}} \Big(\int_0^1 \prod_{\ell \in \mathcal{F}} dw_\ell \Big) \Bigg(\prod_{k=1}^{m} \Big(\prod_{\ell \in \mathcal{F}_k \setminus \mathcal{F}_{k-1}} \frac{\partial}{\partial x_\ell} \Big) \Bigg) f[X^{\mathcal{F}}(w_\ell)],$$

- $\mathcal{J} = (\mathcal{F}_0 \subset \mathcal{F}_1 \cdots \subset \mathcal{F}_{r_{max}} = \mathcal{F})$ *is any partially ordered set of forests $\mathcal{F}_i$ with n vertices.*
- $X^{\mathcal{F}}(w_\ell)$ *is a vector with elements $x_\ell = x_{ij}^{\mathcal{F}}(w_\ell)$:*
  - $x_{ij}^{\mathcal{F}} = 1$ *if $i = j$, or if i and j are connected by $\mathcal{F}_{k-1}$.*
  - $x_{ij}^{\mathcal{F}} = 0$ *if i and j are not connected by $\mathcal{F}_k$,*
  - $x_{ij}^{\mathcal{F}} = \inf_{\ell \in P_{ij}^{\mathcal{F}}} w_\ell$, *if i and j are connected by the forest $\mathcal{F}_k$ but not $\mathcal{F}_{k-1}$, where $P_{ij}^{\mathcal{F}_k}$ is the unique path in the forest that connects i and j,*

The connected functions

- $S_2^c(\lambda) = \sum_n S_{2,n}^c \lambda^n$,

$$S_{2,n}^c \quad = \quad \frac{1}{n!} \sum_{\{\underline{\tau}\},\mathcal{G}^r,\mathcal{T}} \sum_{\mathcal{J}'}{}' \epsilon(\mathcal{J}') \prod_{i'=1}^{n} \int d^3 x_{i'} \delta(x_1)$$
$$\prod_{\ell \in \mathcal{T}} \int_0^1 dw_\ell \, C_{\tau_\ell,\sigma_\ell}(x_\ell, \bar{x}_\ell) \prod_{i=1}^{n} \chi_i(\sigma) \, \mathrm{det}_{\mathrm{left}}(C_j(w)) \; .$$

- $\mathcal{J}' = (\mathcal{F}_0 \subset \mathcal{F}_1 \cdots \subset \mathcal{F}_{r_{max}} = \mathcal{T})$ is called a jungle.
- Perturbation terms are organized into the Gallavotti-Nicolò tree $\mathcal{G}^r$. $r = 2(j + s_+ + s_-)$. $|\mathbf{k}| \sim \gamma^{-r}$.

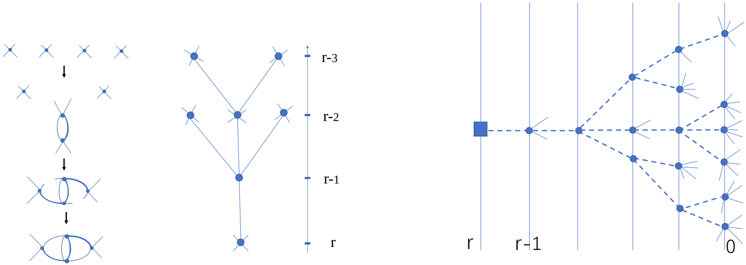

Figure: $r = 2(j + s_+ + s_-)$. $|\mathbf{k}| \sim \gamma^{-r}$.

Difficulties and solutions

- Q3: The dispersion relation receives quantum corrections: $|\Omega(\mathbf{k})|^2 \to |\Omega(\mathbf{k})|^2 + \hat{\Sigma}(k_0, \mathbf{k}, \lambda)$, $\mu \to \mu + \tilde{\delta}\mu(\lambda)$. The interacting Fermi surface is

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  - The cancellations are carried in the multi-scale representation using renormalization theory.

The renormalization of the two-point function

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- Renormalization of the non-local part:

$$\hat{\Sigma}^r_{s_+,s_-}\left[(2\pi T, P_F(\mathbf{k}))_{s_+,s_-}, \hat{\nu}^{\leq r}, \lambda\right] + \hat{\nu}^r_{s_+,s_-}(P_F(\mathbf{k})_{s_+,s_-}, \lambda) = 0.$$

The remainder terms bounded by $\sim \gamma^{-r}$

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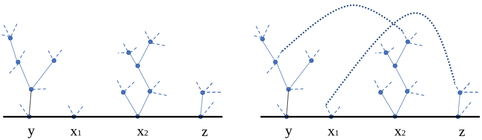
Upper and lower bounds for the self-energy $\Sigma(y, z, \lambda, \beta)$

- The perturbation series of $\Sigma(y, z, \lambda, \beta)$ are labeled by one-particle irreducible graphs (two-connected graphs)
- We partially expand the determinant $\det(\{C(f_i, g_j)\})_{left,\mathcal{T}}$, the multi-arch expansion (Iiagonitzer, Magnen ($\sim$ 1994), Disertori-Rivasseau 2000)



- Establish the upper and lower bounds for the self-energy and its derivatives.

The Hubbard model on the square lattice

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- Theorem (Benffatto, Giuliani, Mastropietro, 2007)
  For $0 < \mu \leq 1$, the ground state is a Fermi liquid for $T \geq T_c$, with $T_c = K_1 \exp(-\frac{C_1}{|\lambda|})$. $K_1, C_1 > 0$.

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- Theorem (ZW, 2022)

  For $\mu = 2 - \mu_0$*,* $\mu_0 \ll 1$ *fixed, the ground state is a not a Fermi liquid for* $T \geq T_c$*, with* $|\lambda \log^2(\mu_0 T)| \leq K_3$*:*

$$T_c = \begin{cases} \frac{K_3}{\mu_0} \exp(-\frac{C_3}{|\lambda|^{1/2}}), & \mu_0 \geq T_c \text{ fixed}, \\ K_4 \exp(-\frac{C_3}{2|\lambda|^{1/2}}), & \mu_0 \to 0. \end{cases} , \quad K_3, C_3 > 0.$$

Conclusions and perspectives

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- Metal-Insulator transitions and many-body localization in Hubbard model.

References

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Thanks for your attention!